The University of Sydney
School of Mathematics and Statistics

# Tutorial for Week 2

MATH1903: Integral Calculus and Modelling (Advanced) — Semester 2, 2009

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## Questions to do in class

**1.** Use the collapsing sum

$$\sum_{j=1}^{n}\left(j^3-(j-1)^3\right)$$

to find a formula for $\sum_{j=1}^{n} j^2$. Adapt the method to find a formula for $\sum_{j=1}^{n} j^3$.

**2.** Let $f(x)=e^x$, and let $P=\{x_0,\dots,x_n\}$ be the partition of $[0,1]$ into $n$ equal parts. Choose sample points $x_j^*=x_j$.

(a) Compute the Riemann sum $\sum_{j=1}^{n} f(x_j^*)\Delta x_j$.

(b) Find the limit of your Riemann sum as $n\to\infty$, and explain why your answer is what it is using a theorem from class.

**3.** Let $f(x)=x^{-2}$, and let $0<a<b$. Let $P=\{x_0,x_1,\dots,x_n\}$ be an arbitrary partition of $[a,b]$, and make the clever choice $x_j^*=\sqrt{x_{j-1}x_j}$. Compute the corresponding Riemann sum. *Hint: Look for a collapsing sum.*

**4.** Suppose that $f(x)$ is monotonically increasing on $[a,b]$, and let $P=\{x_0,x_1,\dots,x_n\}$ be any partition of $[a,b]$ into $n$ subintervals.

(a) Write down expressions for the upper and lower Riemann sums $U_P$ and $L_P$.

(b) Show that $U_P-L_P\le (f(b)-f(a))\|P\|$, where $\|P\|=\max\{\Delta x_1,\dots,\Delta x_n\}$.

**5.** (a) Use the identity $\cos(A-B)-\cos(A+B)=2\sin A\sin B$ to prove that

$$2\sin(j\theta)\sin\left(\tfrac{1}{2}\theta\right)=\cos\left((j-\tfrac{1}{2})\theta\right)-\cos\left((j+\tfrac{1}{2})\theta\right).$$

(b) Deduce that

$$\sum_{j=1}^{n}\sin(j\theta)=\frac{\cos\left(\frac{1}{2}\theta\right)-\cos\left((n+\frac{1}{2})\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)}\quad\text{if } \theta \text{ is not a multiple of } 2\pi.$$

(c) Let $a>0$ and let $\{x_0,\dots,x_n\}$ be a partition of $[0,a]$ into $n$ subintervals of length $a/n$. Let $x_j^*=x_j$ for each $j$. Show that

$$\sum_{j=1}^{n}\sin(x_j^*)\Delta x_j=\frac{a/(2n)}{\sin(a/(2n))}\left[\cos\left(\frac{a}{2n}\right)-\cos\left(a+\frac{a}{2n}\right)\right].$$

Show that this tends to $1-\cos a$ as $n\to\infty$. Explain this using a theorem.

## Challenging problems

**6.** Let $m > 0$ be an integer, and let $0 < a < b$. Use the partition $P = \{a, ar, \ldots, ar^n\}$ with $r = \sqrt[n]{b/a}$ to compute the integral $\int_a^b x^{m-1} dx$ from first principles.

**7.** Suppose that $f$ is an unbounded positive function on the interval $[a, b]$. Show that $f$ is not Riemann integrable on $[a, b]$.

*Hint: Let $M$ be a given (big) number. Show that there is a partition $P$ and sample points $x_j^*$ such that $\|P\|$ small and such that $\sum_{j=1}^n f(x_j^*)\Delta x_j > M$.*

**8.** Suppose that $f$ is continuous on $[a, b]$. Show that if $f \geq 0$ and $\int_a^b f(x)dx = 0$ then $f(x) = 0$ for all $x \in [a, b]$. What happens if we drop the assumption of continuity?

*Hint: Continuity implies that if $f(\alpha) > \epsilon > 0$ for some $\alpha$, then $f(x) > \epsilon/2$ for all $x$ in some (small) interval containing $\alpha$.*