THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Tutorial for Week 2

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2009

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Questions to do in class

1. Use the collapsing sum

$$\sum_{j=1}^{n}\left(j^3-(j-1)^3\right)$$

to find a formula for $\sum_{j=1}^{n} j^2$. Adapt the method to find a formula for $\sum_{j=1}^{n} j^3$.

Solution: Since this is a collapsing sum, we easily compute

$$\sum_{j=1}^{n}\left(j^3-(j-1)^3\right)=n^3.$$

On the other hand we have

$$\sum_{j=1}^{n}\left(j^3-(j-1)^3\right)=\sum_{j=1}^{n}\left(j^3-(j^3-3j^2+3j-1)\right)=3\sum_{j=1}^{n}j^2-3\sum_{j=1}^{n}j+n.$$

We remember that $\sum_{j=1}^{n} j=\frac{n(n+1)}{2}$ (and if we do not remember this, then we use the collapsing sum $\sum_{j=1}^{n}(j^2-(j-1)^2)$ to prove it). Thus the value of $\sum_{j=1}^{n} j^2$ is obtained by solving the equation

$$n^3=3\sum_{j=1}^{n}j^2-\frac{3n(n+1)}{2}+n, \qquad \text{giving} \qquad \sum_{j=1}^{n}j^2=\frac{n(n+1)(2n+1)}{6}.$$

To compute $\sum_{j=1}^{n} j^3$ we look at the collapsing sum

$$n^4=\sum_{j=1}^{n}\left(j^4-(j-1)^4\right)=4\sum_{j=1}^{n}j^3-6\sum_{j=1}^{n}j^2+4\sum_{j=1}^{n}j-n.$$

Using the above formulae for $\sum_{j=1}^{n} j$ and $\sum_{j=1}^{n} j^2$ gives $\sum_{j=1}^{n} j^3=\frac{n^2(n+1)^2}{4}$.

2. Let $f(x)=e^x$, and let $P=\{x_0,\dots,x_n\}$ be the partition of $[0,1]$ into n equal parts. Choose sample points $x_j^*=x_j$.

(a) Compute the Riemann sum $\sum_{j=1}^{n} f(x_j^*)\Delta x_j$.

Solution: Using the geometric sum formula we compute

$$\sum_{j=1}^{n}f(x_j^*)\Delta x_j=\sum_{j=1}^{n}e^{j/n}\frac{1}{n}=(e-1)\frac{n^{-1}}{1-e^{-n^{-1}}}.$$

(b) Find the limit of your Riemann sum as $n \to \infty$, and explain why your answer is what it is using a theorem from class.

Solution: We have

$$\lim_{n\to\infty} \frac{n^{-1}}{1-e^{-n^{-1}}} = \lim_{x\to 0} \frac{x}{1-e^{-x}} = \lim_{x\to 0} \frac{1}{e^{-x}} = 1 \quad \text{(by L'Hôpitals rule)},$$

and so

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(x_j^*)\Delta x_j = e - 1.$$

This is no surprise: Since e^x is continuous we know from lectures that the limit of the Riemann sum must equal the definite integral $\int_0^1 e^x dx$, and we know how to compute this integral; it is $e - 1$.

3. Let $f(x) = x^{-2}$, and let $0 < a < b$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be an arbitrary partition of $[a, b]$, and make the clever choice $x_j^* = \sqrt{x_{j-1}x_j}$. Compute the corresponding Riemann sum. *Hint: Look for a collapsing sum.*

Solution: Notice that $x_{j-1}^2 \le x_{j-1}x_j \le x_j^2$, and so $x_{j-1} \le \sqrt{x_{j-1}x_j} \le x_j$. Thus the sample point $x_j^* = \sqrt{x_{j-1}x_j}$ is indeed between x_{j-1} and x_j. The Riemann sum is

$$\sum_{j=1}^{n} f(x_j^*)\Delta x_j = \sum_{j=1}^{n} \frac{1}{x_{j-1}x_j}(x_j - x_{j-1}) = \sum_{j=1}^{n} \left(\frac{1}{x_{j-1}} - \frac{1}{x_j}\right) = \frac{1}{a} - \frac{1}{b}.$$

Surprisingly, this value does not depend on n.

Notice that by the Fundamental Theorem of Calculus,

$$\int_a^b \frac{1}{x^2}\, dx = -\frac{1}{x}\bigg|_{x=a}^{x=b} = \frac{1}{a} - \frac{1}{b}.$$

We also know that

$$\sum_{j=1}^{n} f(x_j^*)\Delta x_j \to \int_a^b \frac{1}{x^2}\, dx \quad \text{as } \|P\| \to 0$$

(for any partition P of $[a, b]$ and any choice of the x_j^*s). Combining these two things, we knew before making any calculation that $\sum_{j=1}^{n} f(x_j^*)\Delta x_j$ would be close to $1/a - 1/b$. By calculating with this particular choice of x_j^*s, the Riemann sum is not only close, it's actually equal to $1/a - 1/b$.

4. Suppose that $f(x)$ is monotonically increasing on $[a, b]$, and let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a, b]$ into n subintervals.

(a) Write down expressions for the upper and lower Riemann sums U_P and L_P.

Solution: Since f is monotonically increasing, the maximum of f on $[x_{j-1}, x_j]$ occurs at x_j, and the minimum occurs at x_{j-1}. Therefore

$$U_P = \sum_{j=1}^{n} f(x_j)\Delta x_j \qquad \text{and} \qquad L_P = \sum_{j=1}^{n} f(x_{j-1})\Delta x_j.$$

(b) Show that $U_P - L_P \le (f(b) - f(a))\|P\|$, where $\|P\| = \max\{\Delta x_1, \ldots, \Delta x_n\}$.

Solution: Using $\Delta x_j \le \|P\|$ gives

$$U_P - L_P = \sum_{j=1}^{n} (f(x_j) - f(x_{j-1}))\,\Delta x_j \le \|P\| \sum_{j=1}^{n} (f(x_j) - f(x_{j-1}))\,.$$

This last sum is a collapsing sum, and equals $f(b) - f(a)$.

5. (a) Use the identity $\cos(A - B) - \cos(A + B) = 2\sin A \sin B$ to prove that

$$2\sin(j\theta)\sin\left(\tfrac{1}{2}\theta\right) = \cos\left((j - \tfrac{1}{2})\theta\right) - \cos\left((j + \tfrac{1}{2})\theta\right).$$

Solution: The formula $2\sin(j\theta)\sin\left(\frac{1}{2}\theta\right) = \cos\left((j - \frac{1}{2})\theta\right) - \cos\left((j + \frac{1}{2})\theta\right)$ is immediate from $\cos(A-B)-\cos(A+B) = 2\sin A \sin B$, on taking $A = j\theta$ and $B = \frac{1}{2}\theta$. If we write $x_j = \cos\left((j + \frac{1}{2})\theta\right)$, then the above formula can be written, $2\sin(j\theta)\sin\left(\frac{1}{2}\theta\right) = x_{j-1} - x_j$, so that

$$2\sin\left(\tfrac{1}{2}\theta\right)\sum_{j=1}^{n}\sin(j\theta) = \sum_{j=1}^{n}\big(x_{j-1} - x_j\big).$$

The sum on the right collapses to $x_0 - x_n = \cos\left(\frac{1}{2}\theta\right) - \cos\left((n + \frac{1}{2})\theta\right)$.

(b) Deduce that

$$\sum_{j=1}^{n}\sin(j\theta) = \frac{\cos\left(\frac{1}{2}\theta\right) - \cos\left((n + \frac{1}{2})\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)} \quad \text{if } \theta \text{ is not a multiple of } 2\pi.$$

Solution: If we require that θ is not a multiple of 2π, then $\sin\left(\frac{1}{2}\theta\right) \neq 0$. So we can divide both sides of the formula,

$$2\sin\left(\tfrac{1}{2}\theta\right)\sum_{j=1}^{n}\sin(j\theta) = \cos\left(\tfrac{1}{2}\theta\right) - \cos\left((n + \tfrac{1}{2})\theta\right),$$

by $2\sin\left(\frac{1}{2}\theta\right)$, and we get the stated formula for $\sum_{j=1}^{n}\sin(j\theta)$.

(c) Let $a > 0$ and let $\{x_0, \ldots, x_n\}$ be a partition of $[0, a]$ into n subintervals of length a/n. Let $x_j^* = x_j$ for each j. Show that

$$\sum_{j=1}^{n}\sin(x_j^*)\Delta x_j = \frac{a/(2n)}{\sin(a/(2n))}\left[\cos\left(\frac{a}{2n}\right) - \cos\left(a + \frac{a}{2n}\right)\right].$$

Show that this tends to $1 - \cos a$ as $n \to \infty$. Explain this using a theorem.

Solution: If $\{x_0, \ldots, x_n\}$ is the partition of $[0, a]$ into n subintervals of equal length, then $x_j = ja/n$ for each j. So taking $x_j^* = x_j$ for each j, the corresponding Riemann sum is

$$\sum_{j=1}^{n}\sin(x_j^*)\Delta x_j = \sum_{j=1}^{n}\sin\left(\frac{ja}{n}\right)\cdot\frac{a}{n}.$$

This involves the sum $\sum_{j=1}^{n} \sin(j\theta)$ for $\theta = a/n$. Substituting $\theta = a/n$ into our formula above for this sum, we get the stated expression for $\sum_{j=1}^{n} \sin(x_j^*)\Delta x_j$. As $x \to 0$, we know that $(\sin x)/x$ tends to 1. Applying this to $x = a/(2n)$ we see that

$$\frac{a/(2n)}{\sin(a/(2n))} \to 1 \quad \text{as } n \to \infty.$$

Also, by continuity of the cosine function, we know that $\cos\left(\frac{a}{2n}\right) \to \cos 0 = 1$ and that $\cos\left(a + \frac{a}{2n}\right) \to \cos a$ as $n \to \infty$. So the Riemann sum tends to $1 - \cos a$ as $n \to \infty$. This limit was to be expected because $\sum_{j=1}^{n} \sin(x_j^*)\Delta x_j$ tends to $\int_0^a \sin x\, dx$ as $\|P\| \to 0$, and, by the Fundamental Theorem of Calculus, $\int_0^a \sin x\, dx = -\cos x|_0^a = 1 - \cos a$.

Challenging problems

6. Let $m > 0$ be an integer, and let $0 < a < b$. Use the partition $P = \{a, ar, \ldots, ar^n\}$ with $r = \sqrt[n]{b/a}$ to compute the integral $\int_a^b x^{m-1}dx$ from first principles.

Solution: Choose $x_j^* = x_j$. Using the geometric sum formula we see that

$$\sum_{j=1}^{n} f(x_j^*)\Delta x_j = \sum_{j=1}^{n} \left(ar^j\right)^{m-1} \left(ar^j - ar^{j-1}\right) = a^m(r^{mn} - 1)\frac{1 - r^{-1}}{1 - r^{-m}}.$$

Recalling that $r = \sqrt[n]{b/a}$ we get

$$\sum_{j=1}^{n} f(x_j^*)\Delta x_j = (b^m - a^m)\,\frac{1 - \left(\frac{b}{a}\right)^{\frac{1}{n}}}{1 - \left(\frac{b}{a}\right)^{-\frac{m}{n}}}.$$

Let $y = \sqrt[n]{b/a}$. Then as $n \to \infty$ we have $y \to 1$. Therefore (by L'Hôpital's rule)

$$\lim_{n\to\infty} \frac{1 - \left(\frac{b}{a}\right)^{\frac{1}{n}}}{1 - \left(\frac{b}{a}\right)^{-\frac{m}{n}}} = \lim_{y\to 1} \frac{1 - y}{1 - y^{-m}} = \lim_{y\to 1} \frac{1}{my^{-m-1}} = \frac{1}{m}.$$

Thus

$$\sum_{j=1}^{n} f(x_j^*)\Delta x_j \to \frac{1}{m}\left(b^m - a^m\right).$$

Since x^{m-1} is continuous on $[a, b]$ we know that this Riemann sum converges to the definite integral $\int_a^b x^{m-1}dx$, and so $\int_a^b x^{m-1}dx = \frac{1}{m}(b^m - a^m)$.

7. Suppose that f is an unbounded positive function on the interval $[a, b]$. Show that f is not Riemann integrable on $[a, b]$.

Hint: Let M be a given (big) number. Show that there is a partition P and sample points x_j^* such that $\|P\|$ small and such that $\sum_{j=1}^{n} f(x_j^*)\Delta x_j > M$.

Solution: Let P be *any* partition of $[a, b]$. If f were bounded above on each of the n intervals $[x_{j-1}, x_j]$, then it would be bounded above on the whole interval $[a, b]$, contrary to hypothesis. For simplicity of notation, suppose that $f(x)$ is unbounded on $[x_0, x_1]$. Now choose any points $x_j^* \in [x_{j-1}, x_j]$ for $j = 2, 3, \ldots, n$. Let

$$\sum_{j=2}^{n} f(x_j^*)\Delta x_j = K.$$

Because $f(x)$ is unbounded on $[x_0, x_1]$, it is not true that $f(x)(x_1 - x_0) \le M - K$ for all $x \in [x_0, x_1]$. So there must exist $x \in [x_0, x_1]$ such that $f(x)(x_1 - x_0) > M - K$. Let x_1^* be this x. Then

$$f(x_1^*)(x_1 - x_0) > M - K = M - \sum_{j=2}^{n} f(x_j^*)\Delta x_j.$$

Thus

$$\sum_{j=1}^{n} f(x_j^*)\Delta x_j > M.$$

So there is no number A such that the Riemann sums are all close to A whenever $\|P\|$ is small, and hence f is not Riemann integrable on $[a, b]$.

8. Suppose that f is continuous on $[a, b]$. Show that if $f \ge 0$ and $\int_a^b f(x)dx = 0$ then $f(x) = 0$ for all $x \in [a, b]$. What happens if we drop the assumption of continuity?

Hint: Continuity implies that if $f(\alpha) > \epsilon > 0$ for some α, then $f(x) > \epsilon/2$ for all x in some (small) interval containing α.

Solution: If $f(x) \ne 0$ for all $x \in [a, b]$ then there is $\alpha \in [a, b]$ and $\epsilon > 0$ such that $f(\alpha) > \epsilon$. By continuity there is a $\delta > 0$ such that

$$|x - \alpha| \le \delta \implies |f(x) - f(\alpha)| < \epsilon/2.$$

Therefore for $x \in [\alpha - \delta, \alpha + \delta]$ we have

$$f(x) - f(\alpha) > -\epsilon/2, \qquad \text{and so} \qquad f(x) > f(\alpha) - \epsilon/2 > \epsilon - \epsilon/2 = \epsilon/2.$$

Then (since f is positive) we have

$$\int_a^b f(x)dx \ge \int_{\alpha-\delta}^{\alpha+\delta} f(x)dx \ge \int_{\alpha-\delta}^{\alpha+\delta} (\epsilon/2)dx = \frac{\epsilon}{2} \times 2\delta = \epsilon\delta > 0,$$

a contradiction.

Note 1: We have used quite a few "obvious" properties of integrals in the above string of inequalities - at least they are obvious when you think of the Riemann integral as calculating area. You might like to think about how to prove these properties rigorously from the definition of the Riemann integral. Then again, you might like to not do this.

Note 2: In the proof we have assumed for simplicity that $\alpha \in (a, b)$. There are some obvious modifications if $\alpha = a$ or $\alpha = b$.

Finally, if the assumption that f is continuous is dropped then the statement "$\int_a^b f(x)dx = 0$ and $f \ge 0 \implies f(x) = 0$ for all $x \in [a, b]$" fails. For example

$$f(x) = \begin{cases} 0 & \text{if } x \ne \frac{1}{2} \\ 1 & \text{if } x = \frac{1}{2} \end{cases} \quad \text{is Riemann integrable on } [0, 1] \text{ with } \quad \int_0^1 f(x)dx = 0.$$