THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial for Week 3

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2009

Lecturers: Holger Dullin and James Parkinson

Questions to do in class

1. Find the derivative of the following functions.

(a) $f(x) = \displaystyle\int_{-1}^{x} \sqrt{t^3+1}\,dt$

(b) $f(x) = \displaystyle\int_{x}^{4} (2+\sqrt{u})^8\,du$

(c) $f(x) = \displaystyle\int_{1}^{\sqrt{x}} \frac{s^2}{s^2+1}\,ds$

2. Let $f(x) = \int_0^x x\sin(t^2)\,dt$. Find $f''(x)$.

3. Sketch the region bounded by the curves $y = x\sqrt{1-x^2}$ and $y = x - x^3$, and find the area of the region. *Note: The area consists of two crescent-shaped pieces.*

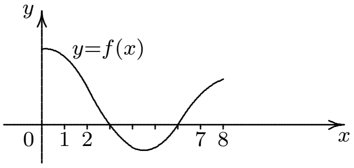
4. The natural logarithm function was defined in lectures by the formula

$$\ln(x) = \int_1^x \frac{1}{t}\,dt \quad \text{for } x > 0.$$

So by The Fundamental Theorem of Calculus we have $\frac{d}{dx}\ln(x) = \frac{1}{x}$ for all $x > 0$, and clearly $\ln(1) = 0$. By differentiating $f(x) = \ln(ax)$ deduce that

$$\ln(ab) = \ln(a) + \ln(b) \quad \text{for all } a, b > 0.$$

5. Suppose that a function $y = f(x)$ has the following graph:



Let $F(x)$ be the function defined by $F(x) = \int_0^x f(t)\,dt$ for $0 \le x \le 8$. Sketch the graph of $y = F(x)$, indicating points where F has a local maximum or minimum, and any points of inflection.

6. If $x\sin(\pi x) = \int_0^{x^2} f(t)\,dt$, find $f(4)$.

Extra problems

7. What is wrong with the following computation: $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-1}^{1} = -2.$

8. Find a function f with continuous derivative f' such that $f(1) = -1$, $f(4) = 7$ and $f'(x) > 3$ for all x, or prove that such a function cannot exist.

9. Find the interval $[a, b]$ which maximises the value of the integral $\displaystyle\int_a^b (2 + x - x^2)\,dx$.

10. Find a function f such that $|x| = \displaystyle\int_0^x f(t)\,dt$ for $-1 \le x \le 1$.

For interest only...

The following question uses a nice mixture of differentiation and integration to show that π, π^2, and e^r ($r \in \mathbb{Q}\backslash\{0\}$) are irrational. It is adapted from proofs given in: *Irrational Numbers*, by Ivan Niven (The Carus Mathematical Monographs, Number 11, 1956).

11. Let $n \ge 0$ be an integer, and let $f_n(x) = \dfrac{x^n(1-x)^n}{n!}$.

(a) Show that $f_n^{(j)}(0)$ and $f_n^{(j)}(1)$ are integers for all $j \in \mathbb{N}$. *Hint: Use the Binomial Theorem to see that $f^{(j)}(0)$ is an integer. Then use $f(1-x) = f(x)$.*

(b) Assume that $\pi^2 = \frac{a}{b}$ is rational, with $a, b \in \mathbb{N}\backslash\{0\}$. Let

$$F_n(x) = b^n \sum_{k=0}^{n} (-1)^k \pi^{2n-2k} f_n^{(2k)}(x)\,.$$

By the previous part we see that $F_n(0)$ and $F_n(1)$ are integers. Calculate $\frac{d}{dx}\left(F_n'(x)\sin\pi x - \pi F_n(x)\cos\pi x\right)$ and deduce that

$$I_n := \pi a^n \int_0^1 f_n(x)\sin\pi x\,dx \qquad \text{is an integer for all } n.$$

(c) Obtain a contradiction by noticing that $0 < f_n(x) < \frac{1}{n!}$ for $x \in (0,1)$. Thus π^2 is irrational. Deduce that π is irrational too.

(d) Let $m \in \mathbb{N}\backslash\{0\}$ and define

$$G_n(x) = \sum_{k=0}^{2n} (-1)^k m^{2n-k} f_n^{(k)}(x)\,.$$

By part (a) we see that $G_n(0)$ and $G_n(1)$ are integers. Calculate $\frac{d}{dx}\left(e^{mx}G_n(x)\right)$ and deduce that

$$m^{2n+1}\int_0^1 e^{mx} f_n(x)dx = e^m G_n(1) - G_n(0)\,.$$

(e) Now assume that $e^m = \frac{p}{q}$ is rational. Obtain a contradiction. Deduce that e^r is irrational for all $r \in \mathbb{Q}\backslash\{0\}$.