THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Tutorial for Week 3

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2009

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Questions to do in class

1. Find the derivative of the following functions.

(a) $f(x) = \displaystyle\int_{-1}^{x} \sqrt{t^3+1}\,dt$

Solution: By the Fundamental Theorem of Calculus, $f'(x) = \sqrt{x^3+1}$.

(b) $f(x) = \displaystyle\int_{x}^{4} (2+\sqrt{u})^8\,du$

Solution: By the Fundamental Theorem of Calculus,

$$f'(x) = -\frac{d}{dx}\int_4^x (2+\sqrt{u})^8\,du = -(2+\sqrt{x})^8.$$

(c) $f(x) = \displaystyle\int_{1}^{\sqrt{x}} \frac{s^2}{s^2+1}\,ds$

Solution: Let $g(x) = \int_1^x \frac{s^2}{s^2+1}\,ds$. Then $f(x) = g(\sqrt{x})$, and so by the Fundamental Theorem of Calculus and the chain rule we compute

$$f'(x) = g'(\sqrt{x})\,\frac{1}{2\sqrt{x}} = \frac{x}{x+1}\,\frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{2(x+1)}.$$

Alternatively, one could set $s = \sqrt{t}$ in the integral to get

$$f(x) = \int_1^x \frac{\sqrt{t}}{2(t+1)}\,dt.$$

Then the result follows directly from the Fundamental Theorem of Calculus.

2. Let $f(x) = \int_0^x x\sin(t^2)\,dt$. Find $f''(x)$.

Solution: Since x is constant as far as the integrating variable t is concerned, we can write $f(x) = x\int_0^x \sin(t^2)\,dt$. Now by the product rule and the Fundamental Theorem of Calculus,

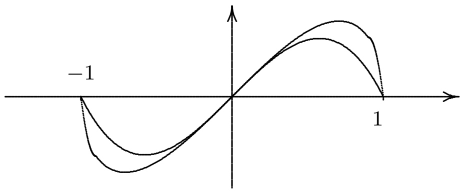
$$f'(x) = x\sin(x^2) + \int_0^x \sin(t^2)\,dt,$$

and

$$f''(x) = \sin(x^2) + x\frac{d}{dx}\sin(x^2) + \sin(x^2) = 2\sin(x^2) + 2x^2\cos(x^2).$$

3. Sketch the region bounded by the curves $y = x\sqrt{1-x^2}$ and $y = x - x^3$, and find the area of the region. *Note: The area consists of two crescent-shaped pieces.*

Solution: The sketch is:



Both $f(x) = x\sqrt{1-x^2}$ and $g(x) = x - x^3$ are odd functions. So the required area is twice the area between the curves in the first quadrant. For $0 \le x \le 1$, we have $0 \le x - x^3 = x(1-x^2) \le x\sqrt{1-x^2}$. So the required area is

$$\begin{aligned} A &= 2\int_0^1 (f(x) - g(x))\, dx \\ &= 2\int_0^1 \left(x\sqrt{1-x^2} - (x - x^3)\right) dx \\ &= 2\left(-\frac{1}{3}(1-x^2)^{3/2} - \left(\frac{1}{2}x^2 - \frac{1}{4}x^4\right)\right)\Bigg|_{x=0}^{x=1} \\ &= \frac{1}{6}. \end{aligned}$$

Note that $\int_{-1}^{1} (f(x) - g(x))\, dx$ gives the wrong answer. Why?

4. The natural logarithm function was defined in lectures by the formula

$$\ln(x) = \int_1^x \frac{1}{t}\, dt \quad \text{for } x > 0.$$

So by The Fundamental Theorem of Calculus we have $\frac{d}{dx}\ln(x) = \frac{1}{x}$ for all $x > 0$, and clearly $\ln(1) = 0$. By differentiating $f(x) = \ln(ax)$ deduce that

$$\ln(ab) = \ln(a) + \ln(b) \quad \text{for all } a, b > 0.$$

Solution: By the chain rule,

$$\frac{d}{dx}\ln(ax) = \frac{1}{ax}a = \frac{1}{x} \quad \text{for all } x > 0.$$

Therefore

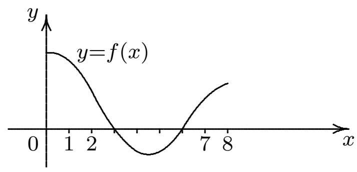
$$\int_1^b \left(\frac{d}{dx}\ln(ax)\right) dx = \int_1^b \frac{1}{x}\, dx = \ln(b).$$

By the Fundamental Theorem of Calculus the left hand side is

$$\ln(ax)\big|_1^b = \ln(ab) - \ln(a),$$

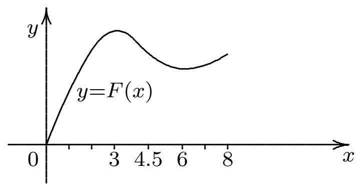
and the result follows.

5. Suppose that a function $y = f(x)$ has the following graph:



Let $F(x)$ be the function defined by $F(x) = \int_0^x f(t)\,dt$ for $0 \le x \le 8$. Sketch the graph of $y = F(x)$, indicating points where F has a local maximum or minimum, and any points of inflection.

Solution: $F'(x) = f(x)$ by the Fundamental Theorem of Calculus, and so we see from the graph of $y = f(x)$ that $F'(x) \ge 0$ for $0 \le x \le 3$ and for $6 \le x \le 8$, while $F'(x) < 0$ for $3 < x < 6$. So $F(x)$ is increasing on $[0, 3]$ and on $[6, 8]$, but decreasing on $[3, 6]$. So $F(x)$ has a local maximum at $x = 3$ and a local minimum at $x = 6$. Also, $F''(x) = f'(x)$, which is negative for $0 < x < 4.5$ and positive for $4.5 < x \le 8$. Hence $F(x)$ is concave downwards on $[0, 4.5]$, concave upwards on $[4.5, 8]$, and has a point of inflection at $x = 4.5$. Note also that $F(0) = 0$. We can now sketch the graph of $y = F(x)$:



6. If $x \sin(\pi x) = \int_0^{x^2} f(t)\,dt$, find $f(4)$.

Solution: If we set $F(x) = \int_0^x f(t)\,dt$, then $F(x^2) = \int_0^{x^2} f(t)\,dt$, and the derivative of this is $2xF'(x^2) = 2xf(x^2)$. So differentiating both sides of the given equation we get

$$\sin(\pi x) + \pi x \cos(\pi x) = 2xf(x^2).$$

Evaluating both sides of this at $x = 2$, we see that $f(4) = \pi/2$.

Extra problems

7. What is wrong with the following computation: $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-1}^{1} = -2.$

Solution: The computation certainly looks worrying, because $\frac{1}{x^2} \geq 0$ yet the integral is claimed to be $-2 < 0$. The problem is that $\frac{1}{x^2}$ has a discontinuity at $x = 0$, which is in the integration range. Therefore there is no reason to expect that the Fundamental Theorem of Calculus holds here.

We will be studying "improper integrals" in class soon. As a preview, we might try to investigate an integral like "$\int_0^1 \frac{1}{x^2} dx$" by first calculating

$$\int_{\epsilon}^{1} \frac{1}{x^2}\,dx$$

(thereby excluding the discontinuity) and then taking $\epsilon \to 0^+$. We have

$$\lim_{\epsilon\to 0^+} \left(\int_{\epsilon}^{1} \frac{1}{x^2}\,dx\right) = \lim_{\epsilon\to 0^+} \left(\frac{1}{\epsilon} - 1\right) = \infty,$$

and so we would say that $\int_0^1 \frac{1}{x^2}\,dx$ "diverges to $+\infty$". As mentioned above, we'll see more of this in lectures soon.

8. Find a function f with continuous derivative f' such that $f(1) = -1$, $f(4) = 7$ and $f'(x) > 3$ for all x, or prove that such a function cannot exist.

Solution: If there is such a function then the Fundamental Theorem of Calculus implies that

$$\int_1^4 f'(x)\,dx = 7 - (-1) = 8.$$

But since $f'(x) > 3$ we have

$$\int_1^4 f'(x)\,dx > \int_1^4 3\,dx = 9,$$

giving a contradiction. So $f(x)$ cannot exist.

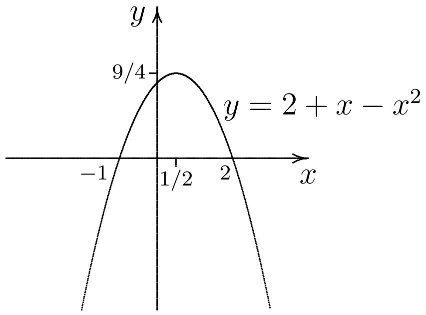
Remark: You might have trouble dreaming up a function f which is differentiable with f' *not* continuous. The standard example is

$$f(x) = \begin{cases} x^2\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

As an exercise you should compute $f'(x)$. There are two cases to consider: If $x \neq 0$ then you just use familiar rules of differentiation, while if $x = 0$ you need to compute the derivative by first principles. You'll see that f is differentiable everywhere. In particular, $f'(0) = 0$, but $\lim_{x\to 0} f(x)$ does not exist. Thus f' is not continuous at $x = 0$.

9. Find the interval $[a, b]$ which maximises the value of the integral $\displaystyle\int_a^b (2 + x - x^2)\, dx$.

Solution: By completing the square we have $2+x-x^2 = \frac{9}{4} - \left(x - \frac{1}{2}\right)^2$, and also we have $2+x-x^2 = -(x+1)(x-2)$. This shows that $2+x-x^2$ has a maximum value of $\frac{9}{4}$, which it attains at $x = \frac{1}{2}$, and that it takes the value 0 at $x = -1$ and at $x = 2$. Of course you could also work these things out using a little calculus if you prefer. Therefore we have the sketch:



Thus, interpreting the integral as area (positive if the graph is above the x-axis, and negative if the graph is below the x-axis) it is clear that the interval that maximises the integral is $[-1, 2]$. Then the integral equals

$$\int_{-1}^{2} (2 + x - x^2)\, dx = \left(2x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_{-1}^{2} = \frac{10}{3} - \left(-\frac{7}{6}\right) = \frac{9}{2}.$$

10. Find a function f such that $|x| = \displaystyle\int_0^x f(t)\, dt$ for $-1 \le x \le 1$.

Solution: The function $y = |x|$ is differentiable at every point except $x = 0$, with

$$\frac{dy}{dx} = \begin{cases} +1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

This suggests that we let

$$f(t) = \begin{cases} +1 & \text{if } t > 0, \\ -1 & \text{if } t < 0, \\ 0 & \text{if } t = 0 \quad \text{(actually } f(0) \text{ can be anything).} \end{cases}$$

This function is continuous everywhere except at 0. If we integrate $f(t)$ from 0 to x, we get x when $x > 0$, which agrees with $|x|$, because $\int_0^x 1\, dt = x$. Similarly, if $x < 0$, then $\int_0^x f(t)\, dt = \int_0^x (-1)\, dt = \int_x^0 1\, dt = -x$, which also agrees with $|x|$. Finally, $\int_0^0 f(t)\, dt = 0 = |0|$. So $\int_0^x f(t)\, dt = |x|$ for all x.

For interest only...

The following question uses a nice mixture of differentiation and integration to show that π, π^2, and e^r ($r \in \mathbb{Q}\backslash\{0\}$) are irrational. It is adapted from proofs given in: *Irrational Numbers*, by Ivan Niven (The Carus Mathematical Monographs, Number 11, 1956).

11. Let $n \geq 0$ be an integer, and let $f_n(x) = \dfrac{x^n(1-x)^n}{n!}$.

(a) Show that $f_n^{(j)}(0)$ and $f_n^{(j)}(1)$ are integers for all $j \in \mathbb{N}$. *Hint: Use the Binomial Theorem to see that $f^{(j)}(0)$ is an integer. Then use $f(1-x) = f(x)$.*

Solution: By the Binomial Theorem we have

$$f_n(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{n+k}}{(n-k)!k!}. \tag{1}$$

If $0 \leq j < n$ then $f^{(j)}(0) = 0$ because $f_n(x) = x^n \times$ (a polynomial).
If $n \leq j \leq 2n$ write $j = n + \nu$ where $0 \leq \nu \leq n$. Then by (1) we have

$$f_n^{(j)}(0) = (-1)^\nu \frac{(n+\nu)!}{(n-\nu)!\nu!} = (-1)^\nu \binom{n}{\nu} (n+1)(n+2)\cdots(n+\nu),$$

which is an integer.
If $j > 2n$ then $f^{(j)}(0) = 0$ (because f is a polynomial of degree $2n$).
Since $f(1-x) = f(x)$ we deduce that $f^{(j)}(1-x) = (-1)^j f^{(j)}(x)$. Therefore $f^{(j)}(1) = (-1)^j f^{(j)}(0)$ is also an integer for all $j \in \mathbb{N}$.

(b) Assume that $\pi^2 = \frac{a}{b}$ is rational, with $a, b \in \mathbb{N}\backslash\{0\}$. Let

$$F_n(x) = b^n \sum_{k=0}^{n} (-1)^k \pi^{2n-2k} f_n^{(2k)}(x)\,.$$

By the previous part we see that $F_n(0)$ and $F_n(1)$ are integers. Calculate $\frac{d}{dx}\left(F_n'(x)\sin\pi x - \pi F_n(x)\cos\pi x\right)$ and deduce that

$$I_n := \pi a^n \int_0^1 f_n(x)\sin\pi x\,dx \qquad \text{is an integer for all } n.$$

Solution: We have

$$\begin{aligned}\frac{d}{dx}\left(F_n'(x)\sin\pi x - \pi F_n(x)\cos\pi x\right) &= b^n\left(F_n''(x) + \pi^2 F_n(x)\right)\sin\pi x \\ &= b^n\left(\pi^{2n+2} f_n(x) + (-1)^n f_n^{(2n+2)}(x)\right)\sin\pi x.\end{aligned}$$

But $f^{(2n+2)}(x) = 0$ for all x, and therefore

$$\frac{d}{dx}\left(F_n'(x)\sin\pi x - \pi F_n(x)\cos\pi x\right) = b^n\pi^{2n+2} f_n(x)\sin\pi x = \pi^2 a^n f_n(x)\sin\pi x.$$

By the Fundamental Theorem of Calculus this implies that

$$\pi^2 a^n \int_0^1 f_n(x)\sin\pi x\,dx = \pi\left(F_n(0) + F_n(1)\right).$$

The result follows since $F_n(0)$ and $F_n(1)$ are integers.

(c) Obtain a contradiction by noticing that $0 < f_n(x) < \frac{1}{n!}$ for $x \in (0,1)$. Thus π^2 is irrational. Deduce that π is irrational too.

Solution: The inequality $0 < f_n(x) < \frac{1}{n!}$ for $x \in (0,1)$ implies that

$$0 < I_n < \pi \frac{a^n}{n!} \int_0^1 \sin \pi x \, dx = \frac{2a^n}{n!}.$$

Since $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ we obtain a contradiction: Once n is large enough we have $0 < I_n < 1$ which contradicts the fact that I_n is an integer.

Therefore π^2 is irrational. If π is rational then π^2 is also rational, and so π is also irrational.

(d) Let $m \in \mathbb{N}\backslash\{0\}$ and define

$$G_n(x) = \sum_{k=0}^{2n} (-1)^k m^{2n-k} f_n^{(k)}(x)\,.$$

By part (a) we see that $G_n(0)$ and $G_n(1)$ are integers. Calculate $\frac{d}{dx}\left(e^{mx} G_n(x)\right)$ and deduce that

$$m^{2n+1} \int_0^1 e^{mx} f_n(x) dx = e^m G_n(1) - G_n(0)\,.$$

Solution: We compute

$$\frac{d}{dx}\left(e^{mx} G_n(x)\right) = m e^{mx} G_n(x) + G_n'(x) = e^{mx}\left(m^{2n+1} f_n(x) + f_n^{(2n+1)}(x)\right)$$

Since $f^{(2n+1)}(x) = 0$ it follows that

$$m^{2n+1} \int_0^1 e^{mx} f_n(x)\, dx = e^m G_n(1) - e^0 G_n(0) = e^m G_n(1) - G_n(0).$$

(e) Now assume that $e^m = \frac{p}{q}$ is rational. Obtain a contradiction. Deduce that e^r is irrational for all $r \in \mathbb{Q}\backslash\{0\}$.

Solution: If $e^m = \frac{p}{q}$ (with $p, q \in \mathbb{N}\backslash\{0\}$) is rational then

$$J_n := q m^{2n+1} \int_0^1 e^{mx} f_n(x)\, dx = p G_n(1) - q G_n(0)$$

is an integer. But $0 < f_n(x) < \frac{1}{n!}$ for $x \in (0,1)$ implies that

$$0 < J_n < \frac{q m^{2n+1}}{n!} \int_0^1 e^{mx}\, dx = \frac{q m^{2n}}{n!}(e^m - 1).$$

We obtain a contradiction as before: For large enough n we have $0 < J_n < 1$, which is impossible since J_n is an integer.

If $r = \frac{m}{k} > 0$ is rational and if e^r is rational, then $e^{rk} = e^m$ is rational. Thus e^r is irrational for all rational $r > 0$, and therefore e^{-r} is irrational too.