THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial for Week 4

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2009

Lecturers: Holger Dullin and James Parkinson

Questions to attempt in class

1. Use cylindrical shells to find the volume of the following:

(a) A cone of height h and circular base of radius r.

(b) A sphere of radius r.

(c) A solid torus obtained by rotating the circle of centre $(R, 0)$ and radius r about the y-axis (assume that $r \leq R$).

2. Find the volume of the solid obtained by:

(a) Rotating about the x-axis the region bounded by the curve $y = a\cosh(x/a)$, the x-axis, and the line $x = b$. Here $a, b > 0$.

(b) Rotating about the y-axis the region bounded by the curve $y = a\cosh(x/a)$, the y-axis, and the line $y = a\cosh(b/a)$. Here $a, b > 0$.

(c) Rotating about the y-axis the region bounded by the curve $y = x\sqrt{1 + x^3}$, the x-axis, and the line $x = 2$.

(d) Rotating about the y-axis the region bounded by the x-axis, the lines $x = a$ and $x = b$, and the curve $y = \sqrt{1 + x^2}$, $a \leq x \leq b$, where $a \geq 0$.

3. Compute the length of:

(a) The parabola $y = x^2$ between $(0, 0)$ and (a, a^2), where $a > 0$.

(b) The graph $y = \ln x$ for $0 < a \leq x \leq b$.

(c) The curve given by $x = a\cos t$, $y = a\sin t$, $z = bt$ with $0 \leq t \leq 2\pi$.

Hint: A change of variables involving $\sinh t$ *will help in (a) and (b). And if you get stuck trying to integrate* $\operatorname{cosech} t$*, try the change of variables* $u = \cosh t$.

4. The surface area of the solid of revolution formed by rotating $y = f(x) \geq 0$, $a \leq x \leq b$, about the x-axis is (not including any end caps):

$$S = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Use this formula to find the surface area of:

(a) A sphere of radius R.

(b) A spheroid obtained by rotating the half ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $y \geq 0$ and $-a \leq x \leq a$ about the x-axis. Be careful with the two cases $a < b$ and $a > b$.

(c) The torus in Question 1(c) above.

Extra questions

5. A *polyhedron* is a closed surface formed by joining a finite number of polygons (faces) edge-to-edge. The polygons need not be regular. Restrict attention to polyhedra that have a well-defined inside and outside. Then the inside together with the boundary forms a solid polyhedron.

(a) Suppose that a particular polyhedron has the property that every face touches a given sphere of radius R tangentially. Prove that the volume V and surface area S of such a polyhedron are related by $V = (1/3)RS$.

Hint: Partition the solid polyhedron into pyramids.

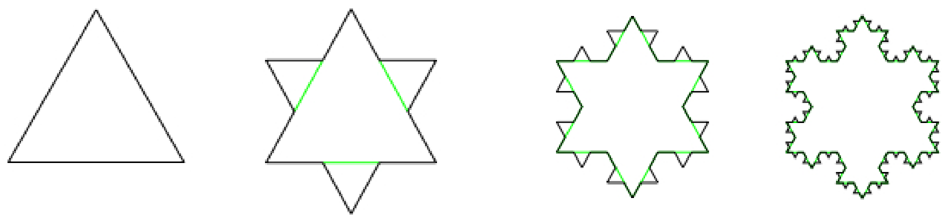
(b) By taking a suitable limit, prove that the sphere has the same property and deduce the surface area of the sphere from its volume.

6. A bowl is in the shape of a hemisphere of radius r cm.

(a) If there is water in the bowl with a depth h at the centre of the bowl, what is the volume of this water?

(b) Suppose that water is poured into the bowl at a constant rate of C cubic centimeters per second. At what rate is the water level rising when $h = r/2$?

7. The *Koch snowflake* is the curve constructed inductively according to the following picture, where the initial equilateral triangle has side length $a > 0$.



At each stage of the construction, each line segment is divided into 3 equal parts and an equilateral triangle is placed on the middle third. The snowflake curve is the "limit curve" of this procedure.

(a) Show that the area of the region enclosed by the snowflake curve is $\frac{2\sqrt{3}}{5}a^2$.

(b) Show that the length of the snowflake curve is infinite.