The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial for Week 4

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2009

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Questions to attempt in class

1. Use cylindrical shells to find the volume of the following:

(a) A cone of height h and circular base of radius r.

Solution: The line segment joining $(0, h)$ to $(r, 0)$ is part of the line with equation $y = h(1 - x/r)$, and so the volume of the cone is

$$V = 2\pi \int_0^r xh\big(1 - \frac{x}{r}\big)\,dx = 2\pi h\left[\frac{x^2}{2} - \frac{x^3}{3r}\right]_0^r = \frac{\pi r^2 h}{3}.$$

(b) A sphere of radius r.

Solution: If V is the volume of the sphere, then we can find $V/2$ by rotating the curve $y = \sqrt{r^2 - x^2}$, $0 \le x \le r$, around the y-axis. Hence

$$\tfrac{1}{2}V = 2\pi \int_0^r x\sqrt{r^2 - x^2}\,dx = \frac{2}{3}\pi r^3.$$

So the volume of the sphere is $V = \frac{4}{3}\pi r^3$.

(c) A solid torus obtained by rotating the circle of centre $(R, 0)$ and radius r about the y-axis (assume that $r \le R$).

Solution: The circle of radius r and centre $(R, 0)$ is given by the equation $(x - R)^2 + y^2 = r^2$. So the top half of the torus is obtained by rotating the curve $y = \sqrt{r^2 - (x - R)^2}$, $R - r \le x \le R + r$, about the y-axis. Hence,

$$\tfrac{1}{2}V = 2\pi \int_{R-r}^{R+r} x\sqrt{r^2 - (x - R)^2}\,dx = 2\pi \int_{-r}^{r} (u + R)\sqrt{r^2 - u^2}\,du,$$

where we have made the substitution $u = x - R$, so that $du = dx$. Break the last integral into

$$2\pi \int_{-r}^{r} (u + R)\sqrt{r^2 - u^2}\,du = 2\pi \int_{-r}^{r} u\sqrt{r^2 - u^2}\,du + 2\pi R \int_{-r}^{r} \sqrt{r^2 - u^2}\,du.$$

The first integral on the right is zero because $u\sqrt{r^2 - u^2}$ is an odd function, and we are integrating over an interval which is symmetric with respect to the origin. Hence the volume of the torus is

$$V = 4\pi R \int_{-r}^{r} \sqrt{r^2 - u^2}\,du = 4\pi R\Big(\frac{\pi r^2}{2}\Big) = 2\pi^2 R r^2.$$

One can see that this is a plausible answer by imagining the torus straightened out into a cylinder. This cylinder would have a circular base of area πr^2 and a height of $2\pi R$.

2. Find the volume of the solid obtained by:

(a) Rotating about the x-axis the region bounded by the curve $y = a\cosh(x/a)$, the x-axis, and the line $x = b$. Here $a, b > 0$.

Solution: Rotating about the x-axis, the volume is found by the disc method to be

$$\begin{aligned} V &= \pi\int_0^b \big(a\cosh(x/a)\big)^2\,dx \\ &= \pi a^2\int_0^b \frac{1+\cosh(2x/a)}{2}\,dx \\ &= \pi a^2\Big[\frac{x}{2}+\frac{a}{4}\sinh(2x/a)\Big]_0^b \\ &= \frac{\pi a^2}{4}\Big(2b + a\sinh(2b/a)\Big). \end{aligned}$$

(b) Rotating about the y-axis the region bounded by the curve $y = a\cosh(x/a)$, the y-axis, and the line $y = a\cosh(b/a)$. Here $a, b > 0$.

Solution: Rotating about the y-axis, the volume swept out by the region between the curve and the x-axis is found by the shell method to be

$$\begin{aligned} &2\pi\int_0^b x\cdot a\cosh(x/a)\,dx \\ &\quad= 2\pi\int_0^{b/a} au\cdot a(\cosh u)\cdot a\,du \qquad \text{(setting } u = x/a\text{)} \\ &\quad= 2\pi a^3\int_0^{b/a} u\cosh u\,du \\ &\quad= 2\pi a^3\int_0^{b/a} u\frac{d}{du}\big(\sinh u\big)\,du \\ &\quad= 2\pi a^3\Big\{\Big[u\sinh u\Big]_0^{b/a} - \int_0^{b/a}\sinh u\,du\Big\} \qquad \text{(integrating by parts)} \\ &\quad= 2\pi a^3\Big\{\frac{b}{a}\sinh\Big(\frac{b}{a}\Big) - \Big[\cosh u\Big]_0^{b/a}\Big\} \\ &\quad= 2\pi a^3\Big\{\frac{b}{a}\sinh\Big(\frac{b}{a}\Big) + 1 - \cosh\Big(\frac{b}{a}\Big)\Big\}. \end{aligned}$$

To get the volume between the surface of revolution and the y-axis, subtract the result just obtained from the volume of a circular cylinder having radius b and height $a\cosh(b/a)$, namely, $\pi ab^2\cosh(b/a)$. The result is

$$V = \pi a(b^2+2a^2)\cosh(b/a) - 2\pi a^2\big\{b\sinh(b/a) + a\big\}.$$

(c) Rotating about the y-axis the region bounded by the curve $y = x\sqrt{1+x^3}$, the x-axis, and the line $x = 2$.

Solution: By the shell method,

$$V = 2\pi\int_0^2 x^2\sqrt{1+x^3}\,dx = \Big[\frac{4\pi}{9}\big(1+x^3\big)^{3/2}\Big]_0^2 = \frac{4\pi}{9}\big(9^{3/2}-1^{3/2}\big) = \frac{104\pi}{9}.$$

(d) Rotating about the y-axis the region bounded by the x-axis, the lines $x = a$ and $x = b$, and the curve $y = \sqrt{1+x^2}$, $a \le x \le b$, where $a \ge 0$.

Solution: We use the shell method. From lectures, the formula for volume is

$$2\pi \int_a^b x f(x)\, dx \;=\; 2\pi \int_a^b x\sqrt{1+x^2}\, dx.$$

Making the substitution $u = 1 + x^2$, we get $\sqrt{1+x^2} = \sqrt{u}$ and $du = 2x\, dx$, and so

$$L \;=\; \pi \int_{1+a^2}^{1+b^2} u^{1/2}\, du \;=\; \left[\frac{2\pi}{3} u^{3/2}\right]_{1+a^2}^{1+b^2} \;=\; \frac{2\pi}{3}\Big\{(1+b^2)^{3/2} - (1+a^2)^{3/2}\Big\}.$$

3. Compute the length of:

(a) The parabola $y = x^2$ between $(0, 0)$ and (a, a^2), where $a > 0$.

Solution: From lectures we know that the length is given by

$$L = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^a \sqrt{1+4x^2}\, dx.$$

Making the change of variable $x = \frac{1}{2}\sinh t$ gives

$$L = \frac{1}{2}\int_0^{\sinh^{-1}(2a)} \sqrt{1+\sinh^2 t}\, \cosh t\, dt = \frac{1}{2}\int_0^{\sinh^{-1}(2a)} \cosh^2 t\, dt.$$

The formulas $\cosh^2 t - \sinh^2 t = 1$ and $\cosh^2 t + \sinh^2 t = \cosh(2t)$ imply that $\cosh^2 t = \frac{1}{2}(1 + \cosh(2t))$, and so

$$L = \frac{1}{4}\int_0^{\sinh^{-1}(2a)} (1 + \cosh(2t))\; dt = \frac{1}{4}\sinh^{-1}(2a) + \frac{1}{8}\sinh(2\sinh^{-1}(2a)).$$

Since $\sinh(2y) = 2\sinh y \cosh y = 2\sinh y\sqrt{1+\sinh^2 y}$ we have

$$L = \frac{1}{4}\sinh^{-1}(2a) + \frac{1}{4}(2a)\sqrt{1+(2a)^2} = \frac{1}{4}\sinh^{-1}(2a) + \frac{a}{2}\sqrt{1+4a^2}.$$

This can be simplified further: If $x = \sinh^{-1} y$ then $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$. Rearranging gives $(e^x)^2 - 2y(e^x) - 1 = 0$, and so $e^x = y \pm \sqrt{1+y^2}$. Since $e^x > 0$ we must take the $+$ sign, and hence $x = \ln(y + \sqrt{1+y^2})$. It follows that

$$L = \frac{1}{4}\ln(2a + \sqrt{1+4a^2}) + \frac{a}{2}\sqrt{1+4a^2}.$$

(b) The graph $y = \ln x$ for $0 < a \le x \le b$.

Solution: The required length is

$$L = \int_a^b \sqrt{1 + \frac{1}{x^2}} dx = \int_a^b \frac{\sqrt{1+x^2}}{x}\, dx.$$

Setting $x = \sinh t$ gives

$$L = \int_{\sinh^{-1}(a)}^{\sinh^{-1}(b)} \frac{\cosh^2 t}{\sinh t}\,dt = \int_{\sinh^{-1}(a)}^{\sinh^{-1}(b)} \frac{1}{\sinh t}\,dt + \int_{\sinh^{-1}(a)}^{\sinh^{-1}(b)} \sinh t\,dt.$$

The second integral equals

$$\cosh(\sinh^{-1}(b)) - \cosh(\sinh^{-1}(a)) = \sqrt{1+b^2} - \sqrt{1+a^2},$$

and setting $u = \cosh t$ gives

$$\int_{\sinh^{-1}(a)}^{\sinh^{-1}(b)} \frac{1}{\sinh t}\,dt = \int_{\sqrt{1+a^2}}^{\sqrt{1+b^2}} \frac{1}{u^2-1}du = \frac{1}{2}\ln\frac{(\sqrt{1+b^2}-1)(\sqrt{1+a^2}+1)}{(\sqrt{1+b^2}+1)(\sqrt{1+a^2}-1)}.$$

Thus

$$\begin{aligned} L &= \sqrt{1+b^2} - \sqrt{1+a^2} + \frac{1}{2}\ln\frac{(\sqrt{1+b^2}-1)(\sqrt{1+a^2}+1)}{(\sqrt{1+b^2}+1)(\sqrt{1+a^2}-1)} \\ &= \sqrt{1+b^2} - \sqrt{1+a^2} + \ln\frac{a(1+\sqrt{1+a^2})}{b(1+\sqrt{1+b^2})}. \end{aligned}$$

(c) The curve given by $x = a\cos t$, $y = a\sin t$, $z = bt$ with $0 \le t \le 2\pi$.

Solution: From lectures the length is given by

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt \\ &= \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2}\,dt \\ &= \int_0^{2\pi} \sqrt{a^2+b^2}\,dt \\ &= 2\pi\sqrt{a^2+b^2}\,. \end{aligned}$$

Hint: A change of variables involving $\sinh t$ *will help in (a) and (b). And if you get stuck trying to integrate* $\operatorname{cosech} t$*, try the change of variables* $u = \cosh t$.

4. The surface area of the solid of revolution formed by rotating $y = f(x) \ge 0$, $a \le x \le b$, about the x-axis is (not including any end caps):

$$S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

Use this formula to find the surface area of:

(a) A sphere of radius R.

Solution: The sphere is obtained by rotating the semicircle $y = \sqrt{R^2 - x^2}$, $-R \le x \le R$, about the x-axis. Hence the surface area of the sphere is

$$S = 2\pi\int_{-R}^{R} \sqrt{R^2-x^2}\sqrt{1+\frac{x^2}{R^2-x^2}}\,dx = 2\pi\int_{-R}^{R} R\,dx = 4\pi R^2.$$

(b) A spheroid obtained by rotating the half ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $y \geq 0$ and $-a \leq x \leq a$ about the x-axis. Be careful with the two cases $a<b$ and $a>b$.

Solution: Using the formula, the surface area of the spheroid is

$$S \;=\; \frac{2\pi b}{a^2}\int_{-a}^{a}\sqrt{a^4-(a^2-b^2)x^2}\,dx \;=\; \frac{4\pi b}{a^2}\int_0^a \sqrt{a^4-(a^2-b^2)x^2}\,dx.$$

To evaluate this integral we need to treat the cases $a>b$ and $a<b$ separately. So suppose first that $a>b$. Then the integral is

$$S=\frac{4\pi b}{R}\int_0^a \sqrt{R^2-x^2}\,dx, \qquad \text{where} \qquad R=\frac{a^2}{\sqrt{a^2-b^2}}.$$

The integral is computed by setting $x=R\sin\theta$:

$$\begin{aligned}\int_0^a \sqrt{R^2-x^2}\,dx &= R^2\int_0^{\sin^{-1}(a/R)} \cos^2\theta\,d\theta\\ &= \frac{R^2}{2}\int_0^{\sin^{-1}(a/R)} (1+\cos 2\theta)\,d\theta\\ &= \frac{R^2}{2}\sin^{-1}\left(\frac{a}{R}\right)+\frac{R^2}{4}\sin\left(2\sin^{-1}(a/R)\right)\\ &= \frac{R^2}{2}\sin^{-1}\left(\frac{a}{R}\right)+\frac{a}{2}\sqrt{R^2-a^2}.\end{aligned}$$

Therefore

$$S=2\pi bR\sin^{-1}\left(\frac{a}{R}\right)+\frac{2\pi ab}{R}\sqrt{R^2-a^2}.$$

Remembering the formula for R and simplifying gives

$$S=2\pi b^2+\frac{2\pi a^2 b}{\sqrt{a^2-b^2}}\cos^{-1}\left(\frac{b}{a}\right).$$

If $a<b$ then the integral is

$$S=\frac{4\pi b}{R}\int_0^a \sqrt{R^2+x^2}\,dx, \qquad \text{where} \qquad R=\frac{a^2}{\sqrt{b^2-a^2}}.$$

Setting $x=R\sinh\theta$, and making a computation analogous to above, we see that

$$S=2\pi bR\sinh^{-1}\left(\frac{a}{R}\right)+\frac{2\pi ab}{R}\sqrt{R^2+a^2},$$

which simplifies to

$$S \;=\; 2\pi b^2 \;+\; \frac{2\pi a^2 b}{\sqrt{b^2-a^2}}\cosh^{-1}\left(\frac{b}{a}\right).$$

(c) The torus in Question 1(c) above.

Solution: Place the circular cross-section on the y-axis at $(0,R)$ so that its equation is $x^2+(y-R)^2=r^2$. Then consider separately the surfaces swept out by rotating the upper semicircle and the lower semicircle about the x-axis. The surface area swept out by the upper semicircle is

$$\begin{aligned} S_1 &= 2\pi \int_{-r}^{r} \left\{ \frac{Rr}{\sqrt{r^2 - x^2}} + r \right\} dx \\ &= 4\pi \left[Rr \sin^{-1}\left(\frac{x}{r}\right) + rx \right]_0^r \\ &= 2\pi^2 Rr + 4\pi r^2. \end{aligned}$$

Similarly the surface swept out by the lower semicircle is

$$\begin{aligned} S_2 &= 2\pi \int_{-r}^{r} \left\{ \frac{Rr}{\sqrt{r^2 - x^2}} - r \right\} dx \\ &= 4\pi \left[Rr \sin^{-1}\left(\frac{x}{r}\right) - rx \right]_0^r \\ &= 2\pi^2 Rr - 4\pi r^2. \end{aligned}$$

Therefore the total surface area is $S = S_1 + S_2 = 4\pi^2 Rr$.

Extra questions

5. A *polyhedron* is a closed surface formed by joining a finite number of polygons (faces) edge-to-edge. The polygons need not be regular. Restrict attention to polyhedra that have a well-defined inside and outside. Then the inside together with the boundary forms a solid polyhedron.

(a) Suppose that a particular polyhedron has the property that every face touches a given sphere of radius R tangentially. Prove that the volume V and surface area S of such a polyhedron are related by $V = (1/3)RS$.

Hint: Partition the solid polyhedron into pyramids.

Solution: Consider one of the polygonal faces and join every point of that face to the centre of the sphere by a straight line. The solid so formed is a pyramid having the polygon as its base. Since the face touches the sphere, the radius ending at the point of contact is perpendicular to the face. Hence R is the perpendicular height of the pyramid. Let A be the area of the face. Then the volume of the pyramid is $(1/3)AR$.

Next, the totality of such pyramids formed on all the faces fills up the solid polyhedron. All the pyramids have the same perpendicular height, namely, the radius R of the sphere. Hence the total volume is $(1/3)R$ times the total area of the faces. In other words, $V = (1/3)RS$, as required.

(b) By taking a suitable limit, prove that the sphere has the same property and deduce the surface area of the sphere from its volume.

Solution: The sphere itself can be approached arbitrarily closely by polyhedra of the above type in such a way that the number of faces tends to infinity while the largest face (measured by its longest diagonal or edge) tends to infinitesimal size. Since the formula, $V = (1/3)RS$, is exact for all the approximating polyhedra, it holds also for the limiting sphere. Given that $V = (4/3)\pi R^3$, we conclude that $S = 4\pi R^2$.

6. A bowl is in the shape of a hemisphere of radius r cm.

(a) If there is water in the bowl with a depth h at the centre of the bowl, what is the volume of this water?

Solution: The bowl can be formed by rotating a suitable circular quadrant about the y-axis. Let the centre be on the y-axis at $(0, r)$. Then the equation of the quadrant is $x^2 + (y-r)^2 = r^2$, the relevant piece running from $(0, 0)$ to (r, r). We wish to use the disc method to calculate volume. Hence we need x expressed as a function of y. The required function is $x = \sqrt{r^2 - (y-r)^2} = \sqrt{2ry - y^2}$ for $0 \le y \le r$. So the volume of the water is the same as the volume obtained by rotating this curve around the y-axis between $y = 0$ and $y = h$. Thus,

$$V = \pi \int_0^h \big(2ry - y^2\big)\, dy = \pi\big(rh^2 - \tfrac{1}{3}h^3\big) = \tfrac{1}{3}\pi h^2(3r - h).$$

(b) Suppose that water is poured into the bowl at a constant rate of C cubic centimeters per second. At what rate is the water level rising when $h = r/2$?

Solution: By part (a),

$$V = \tfrac{1}{3}\pi h^2(3r-h) = \pi r h^2 - \tfrac{1}{3}\pi h^3.$$

We are given that $dV/dt = C$. So

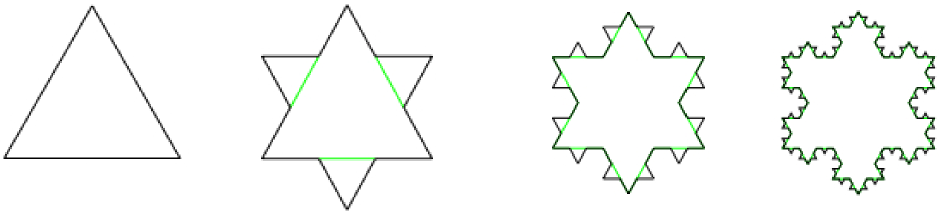
$$C = \frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} = (2\pi r h - \pi h^2)\frac{dh}{dt}.$$

Therefore,

$$\frac{dh}{dt} = \frac{C}{\pi h(2r-h)}.$$

When $h = r/2$, the water level is rising at the rate $dh/dt = 4C/(3\pi r^2)$.

7. The *Koch snowflake* is the curve constructed inductively according to the following picture, where the initial equilateral triangle has side length $a > 0$.



At each stage of the construction, each line segment is divided into 3 equal parts and an equilateral triangle is placed on the middle third. The snowflake curve is the "limit curve" of this procedure.

(a) Show that the area of the region enclosed by the snowflake curve is $\frac{2\sqrt{3}}{5}a^2$.

Solution: Let the sequence of curves in the construction of the snowflake curve be $C_0, C_1, C_2, \ldots$, so that C_0 is the initial equilateral triangle with side length $a > 0$. We first compute the number of line segments on C_k. We have the recursion formula

$$(\text{number of line segments on } C_k) = 4 \times (\text{number of line segments on } C_{k-1}).$$

To see this, note that in the construction of C_k from C_{k-1} each segment of C_{k-1} is broken into 3 segments. When the new equilateral triangle is placed on the middle segment we cover up the middle segment, but add two new segments from the other two sides of the triangle. Thus each segment of C_{k-1} produces 4 segments in C_k. Hence

$$(\#\text{ of segments on } C_k) = 4^k \times (\#\text{ of segments on } C_0) = 3 \times 4^k.$$

We can now derive a recursive formula for the area A_k enclosed by C_k. We have:

$$\begin{aligned} A_k &= A_{k-1} + (\text{area of new triangles}) \\ &= A_{k-1} + (\text{number of new triangles}) \times (\text{area of triangle}). \end{aligned}$$

The number of new triangles equals the number of line segments on C_{k-1}, and each new triangle added to C_{k-1} to make C_k has side length $3^{-k}a$. Therefore

$$A_k = A_{k-1} + \frac{\sqrt{3}}{12}\left(\frac{4}{9}\right)^{k-1} a^2.$$

It follows that

$$A_k = A_0 + \frac{\sqrt{3}}{12}a^2\left(\left(\frac{4}{9}\right)^{k-1} + \left(\frac{4}{9}\right)^{k-2} + \cdots + 1\right).$$

Using the geometric sum formula we get

$$A_k = \frac{\sqrt{3}}{4}a^2 + \frac{3\sqrt{3}}{20}a^2\left(1 - \left(\frac{4}{9}\right)^k\right) = \frac{2\sqrt{3}}{5}a^2 - \frac{3\sqrt{3}}{20}\left(\frac{4}{9}\right)^k a^2.$$

The area enclosed by the snowflake curve is given by taking the limit as $k \to \infty$, and thus $A = \frac{2\sqrt{3}}{5}a^2$.

(b) Show that the length of the snowflake curve is infinite.

Solution: The length of the curve C_k is

$$L_k = (\text{number of line segments on } C_k) \times (\text{length of each segment}).$$

The number of line segments is 3×4^k, and the length of each segment is $3^{-(k+1)}a$. Therefore

$$L_k = \left(\frac{4}{3}\right)^k a,$$

which tends to infinity as $k \to \infty$.

Remark: The snowflake curve is continuous everywhere but differentiable nowhere.