The University of Sydney
School of Mathematics and Statistics

Tutorial for Week 5

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2009

Lecturers: Holger Dullin and James Parkinson

Questions to attempt in class

1. Determine whether the following improper integrals exist by evaluating an appropriate limit of a (proper) integral.

(a) $\displaystyle\int_{\pi/4}^{\pi/2} \sec^2 x\, dx$ (c) $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\, dx$

(b) $\displaystyle\int_0^1 \frac{\ln x}{x^{1/3}}\, dx$ (d) $\displaystyle\int_1^{\infty} \sin(\pi x)\, dx$

2. Determine whether the following improper integrals exist. *Hint: Comparison Test.*

(a) $\displaystyle\int_0^1 \frac{e^{-x}}{x}\, dx$ (c) $\displaystyle\int_1^{\infty} \frac{\cos^2 x}{x^2}\, dx$

(b) $\displaystyle\int_1^{\infty} \frac{e^{-x}}{\sqrt{x}}\, dx$ (d) $\displaystyle\int_0^{\infty} x^3 e^{-x}\, dx$

3. Use integration by parts to find reduction formulae for the following integrals.

(a) $\displaystyle\int \cos^n \theta\, d\theta$ (b) $\displaystyle\int (\ln x)^n\, dx$

4. Use the results of the previous question to evaluate the following integrals.

(a) $\displaystyle\int \cos^5 \theta\, d\theta$ (b) $\displaystyle\int_0^1 (\ln x)^n\, dx$

5. Use integration by parts to show that

$$\int_1^b \frac{\sin x}{x}\, dx = \cos(1) - \frac{\cos b}{b} - \int_1^b \frac{\cos x}{x^2}\, dx.$$

Show that the improper integral $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\, dx$ exists. See Question 10.

6. Make the substitution $x = \pi - t$ to show that if f is continuous then

$$\int_0^{\pi} x f(\sin x)\, dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x)\, dx.$$

Extra questions

7. Find a reduction formula for the integral $I_n = \dfrac{1}{n!}\displaystyle\int_0^1 (1-x^2)^n \cos\big(\tfrac{1}{2}\pi x\big)\,dx.$

8. Let $x \in \mathbb{R}$, and let $n \geq 0$ be an integer.

(a) Use a reduction formula to prove that

$$\int_0^x (x-t)^n e^t\,dt = n!\left(e^x - \sum_{k=0}^{n} \frac{x^k}{k!}\right).$$

(b) Set $x = 1$ in (a) and deduce that e is irrational.
Hint: If e is rational then the integral is an integer for sufficiently large n.

(c) Use (a) to show that

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!} = e^x \qquad \text{for all } x \in \mathbb{R}.$$

9. For $n \geq 0$ let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n\theta\,d\theta.$

(a) Derive a reduction formula for I_n, and use it to deduce that

$$I_{2n} = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2} \qquad \text{and} \qquad I_{2n+1} = \frac{(2n)!!}{(2n+1)!!},$$

where $(2n)!! = 2\cdot 4\cdots(2n)$ and $(2n+1)!! = 1\cdot 3\cdots(2n+1)$.

(b) Show that $I_{2n-1} \leq I_{2n} \leq I_{2n+1}$, and deduce that that

$$1 \leq \frac{1\cdot 3\cdot 3\cdots(2n-1)(2n-1)}{2\cdot 2\cdot 4\cdots(2n-2)(2n)}\frac{\pi}{2} \leq \frac{2n}{2n+1}.$$

Hence prove the *Wallis Product Formula* for π:

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7}\cdots = \lim_{n\to\infty}\frac{2^{4n}n!^4}{2n(2n)!^2}.$$

10. For integers $n \geq 1$ let

$$a_n = \int_0^{\frac{\pi}{2}} \sin(2nx)\cot x\,dx \qquad \text{and} \qquad b_n = \int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{x}\,dx.$$

(a) Prove the formula

$$\frac{\sin(2nx)}{\sin x} = 2\sum_{k=1}^{n}\cos\left((2k-1)x\right) \qquad \text{for } x \notin \pi\mathbb{Z} \text{ and } n \geq 1.$$

(b) Deduce that $a_n = \frac{\pi}{2}$ for all $n \geq 1$.

(c) Use integration by parts to show that $\lim_{n\to\infty}(a_n - b_n) = 0$.

(d) Deduce that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$