THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Tutorial for Week 5

MATH1903: Integral Calculus and Modelling (Advanced) — Semester 2, 2009

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Questions to attempt in class

1. Determine whether the following improper integrals exist by evaluating an appropriate limit of a (proper) integral.

(a) $\displaystyle\int_{\pi/4}^{\pi/2} \sec^2 x\,dx$

Solution: Since $\sec^2 x = 1/\cos^2 x \to \infty$ as $x \to \pi/2$, the integrand is unbounded, and the integral is improper. If $0 < \epsilon < \pi/4$,

$$\begin{aligned}\int_{\pi/4}^{\pi/2-\epsilon} \sec^2 x\,dx &= \int_{\pi/4}^{\pi/2-\epsilon} \frac{d}{dx}\big(\tan x\big)\,dx\\ &= \tan(\pi/2-\epsilon) - \tan(\pi/4)\\ &= \tan(\pi/2-\epsilon) - 1\\ &\to \infty \quad \text{as } \epsilon \to 0.\end{aligned}$$

So the improper integral does not exist. More precisely we say that the integral diverges to $+\infty$.

(b) $\displaystyle\int_0^1 \frac{\ln x}{x^{1/3}}\,dx$

Solution: As $x \to 0^+$ we see that $(\ln x)/x^{1/3} \to -\infty$. So the integral is improper. If $0 < \epsilon < 1$, then

$$\begin{aligned}\int_\epsilon^1 \frac{\ln x}{x^{1/3}}\,dx &= \Big[\ln x\,\frac{3}{2}x^{2/3}\Big]_{x=\epsilon}^{x=1} - \frac{3}{2}\int_\epsilon^1 x^{2/3}\,\frac{1}{x}\,dx \quad \text{(integrating by parts)}\\ &= -\frac{3}{2}\epsilon^{2/3}\ln\epsilon - \frac{9}{4}\big(1-\epsilon^{2/3}\big).\end{aligned}$$

As $\epsilon \to 0^+$, by L'Hôpital's Rule $\epsilon^c \ln\epsilon \to 0$ for any fixed $c > 0$. Therefore

$$\int_\epsilon^1 \frac{\ln x}{x^{1/3}}\,dx \to -\frac{9}{4} \quad \text{as } \epsilon \to 0.$$

So the improper integral exists and equals $-9/4$.

(c) $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$

Solution: Integrating by parts gives

$$\int_1^N \frac{\ln x}{x^2}\,dx = -\frac{\ln}{x}\Big|_1^N + \int_1^N \frac{1}{x^2}\,dx = -\frac{\ln N}{N} + 1 - N^{-1}.$$

Since $\lim_{N\to\infty} \frac{\ln N}{N} = 0$ we see that $\lim_{N\to\infty} \int_1^N \frac{\ln x}{x^2}\,dx = 1$. Therefore the improper integral exists, and equals 1.

(d) $\displaystyle\int_1^\infty \sin(\pi x)\,dx$

Solution: We compute

$$\int_1^b \sin(\pi x)\,dx = -\frac{\cos b\pi}{\pi} - \frac{1}{\pi}.$$

Since $\lim_{b\to\infty} \cos b\pi$ does not exist we see that the improper integral does not exist. Note that it does not diverge to ∞; rather $\int_1^b \cos(\pi x)\,dx$ oscillates, taking values between 0 and $-2/\pi$.

2. Determine whether the following improper integrals exist. *Hint: Comparison Test.*

(a) $\displaystyle\int_0^1 \frac{e^{-x}}{x}\,dx$

Solution: Because $e^{-x}/x \sim 1/x$ for x near 0, we expect to get divergence by comparison with the integral of $1/x$ on $[0,1]$. Since $e^{-x} \geq 1/e$ for all $x \in [0,1]$ we have $\frac{e^{-x}}{x} \geq \frac{1/e}{x}$ on $(0,1]$. Now

$$\int_0^1 \frac{1/e}{x}\,dx \;=\; \lim_{\epsilon\to 0} \frac{1}{e}\int_\epsilon^1 \frac{1}{x}\,dx \;=\; \frac{1}{e}\lim_{\epsilon\to 0} \ln\Big(\frac{1}{\epsilon}\Big) \;=\; +\infty.$$

So $\int_0^1 \frac{e^{-x}}{x}\,dx$ diverges to $+\infty$ too, by the Comparison Test. In other words, the improper integral does not exist.

(b) $\displaystyle\int_1^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$

Solution: $0 \leq e^{-x}/\sqrt{x} \leq e^{-x}$ for all $x \geq 1$. Also,

$$\int_1^b e^{-x}\,dx = \Big[-e^{-x}\Big]_1^b = e^{-1} - e^{-b} \to e^{-1} \quad \text{as } b \to \infty.$$

So $\int_1^\infty e^{-x}dx$ converges, so $\int_1^\infty \frac{e^{-x}}{\sqrt{x}}dx$ converges too by the Comparison Test.

(c) $\displaystyle\int_1^\infty \frac{\cos^2 x}{x^2}\,dx$

Solution: Use

$$0 \leq \frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}\,,$$

and the fact (shown in lectures) that $\int_1^\infty \frac{1}{x^2}\,dx$ converges. Now the Comparison Test shows that the given improper integral also exists.

(d) $\displaystyle\int_0^\infty x^3 e^{-x}\,dx$

Solution: Note first that $x^3/e^x \to 0$ as $x \to \infty$, as you can see using L'Hôpital's Rule, for example. So there is a number M such that $x^3/e^x \leq 1$ once $x \geq M$. Replacing x by $x/2$, we see that $(x/2)^3/e^{x/2} \leq 1$ once $x/2 \geq M$. That is, $x^3 \leq 8e^{x/2}$ once $x \geq 2M$. So the integrand $x^3 e^{-x}$ may be estimated as follows:

$$x^3 e^{-x} \;\leq\; 8e^{x/2}e^{-x} \;=\; 8e^{-x/2} \quad \text{once } x \geq 2M.$$

Now $\int_{2M}^{\infty} 8e^{-x/2}\,dx$ converges, by an easy calculation. So $\int_{2M}^{\infty} x^3e^{-x}\,dx$ converges by the Comparison Test. For $0 \le x \le 2M$, x^3e^{-x} is continuous, and so $\int_0^{2M} x^3e^{-x}\,dx$ exists. Hence

$$\int_0^{\infty} x^3e^{-x}\,dx = \int_0^{2M} x^3e^{-x}\,dx + \int_{2M}^{\infty} x^3e^{-x}\,dx$$

exists. Using integration by parts, it is easy to calculate its value exactly: it equals 6.

If you prefer to avoid breaking the integral up into two parts as above, you could instead argue as follows: By Calculus, we find that $x^3e^{-x/2}$ takes its maximum value of $C = 216e^{-3}$ at $x = 6$. Thus $x^3 \le Ce^{x/2}$ for all $x \ge 0$. Hence $x^3e^{-x} \le Ce^{-x/2}$ for all $x \ge 0$. Since $\int_0^{\infty} Ce^{-x/2}\,dx$ converges by an easy direct calculation, so does $\int_0^{\infty} x^3e^{-x}\,dx$, by the Comparison Test.

3. Use integration by parts to find reduction formulae for the following integrals.

(a) $\displaystyle\int \cos^n\theta\,d\theta$

Solution: Let $u = \cos^{n-1}\theta$ and $\frac{dv}{d\theta} = \cos\theta$. Then

$$\begin{aligned}\int \cos^n\theta\,d\theta &= \cos^{n-1}\theta\sin\theta + (n-1)\int \cos^{n-2}\theta\sin^2\theta\,d\theta \\ &= \cos^{n-1}\theta\sin\theta + (n-1)\int \cos^{n-2}\theta\,d\theta - (n-1)\int \cos^n\theta\,d\theta,\end{aligned}$$

Rearranging the last expression gives the reduction formula:

$$\int \cos^n\theta\,d\theta = \frac{1}{n}\cos^{n-1}\theta\sin\theta + \frac{n-1}{n}\int \cos^{n-2}\theta\,d\theta.$$

(b) $\displaystyle\int (\ln x)^n\,dx$

Solution: Let $u = (\ln x)^n$ and $\frac{dv}{dx} = 1$. Then

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx.$$

4. Use the results of the previous question to evaluate the following integrals.

(a) $\displaystyle\int \cos^5\theta\,d\theta$

Solution: From the reduction formula,

$$\int \cos^n\theta\,d\theta = \frac{1}{n}\cos^{n-1}\theta\sin\theta + \frac{n-1}{n}\int \cos^{n-2}\theta\,d\theta,$$

we get

$$\begin{aligned}\int \cos^3\theta\,d\theta &= \frac{1}{3}\cos^2\theta\sin\theta + \frac{2}{3}\int \cos\theta\,d\theta \\ &= \frac{1}{3}\cos^2\theta\sin\theta + \frac{2}{3}\sin\theta + C_1\,.\end{aligned}$$

Then,

$$\begin{aligned}
\int \cos^5\theta\,d\theta &= \frac{1}{5}\cos^4\theta\sin\theta + \frac{4}{5}\int\cos^3\theta\,d\theta \\
&= \frac{1}{5}\cos^4\theta\sin\theta + \frac{4}{5}\Big\{\frac{1}{3}\cos^2\theta\sin\theta + \frac{2}{3}\sin\theta + C_1\Big\} \\
&= \frac{1}{5}\cos^4\theta\sin\theta + \frac{4}{15}\cos^2\theta\sin\theta + \frac{8}{15}\sin\theta + C.
\end{aligned}$$

(b) $\displaystyle\int_0^1 (\ln x)^n\,dx$

Solution: let $J_n = \int_0^1 (\ln x)^n\,dx = \lim_{\epsilon\to 0^+}\int_\epsilon^1 (\ln x)^n\,dx$. Then, for $n \geq 1$, the reduction formula in the previous question gives

$$\begin{aligned}
J_n &= \int_0^1 (\ln x)^n\,dx \\
&= \lim_{\epsilon\to 0^+}\Big[x(\ln x)^n\Big]_\epsilon^1 - n\int_0^1 (\ln x)^{n-1}\,dx \\
&= -nJ_{n-1}\,.
\end{aligned}$$

The limit can be proved by applying l'Hôpital's rule n times to the ratio $(\ln\epsilon)^n/(\epsilon^{-1})$. Now, repeated application of the reduction formula for J_n gives

$$J_n = -nJ_{n-1} = n(n-1)J_{n-2} = -n(n-1)(n-2)J_{n-3} = \cdots = (-1)^n n! J_0.$$

A trivial integration gives $J_0 = 1$. Hence,

$$J_n = (-1)^n n!\,.$$

5. Use integration by parts to show that

$$\int_1^b \frac{\sin x}{x}\,dx = \cos(1) - \frac{\cos b}{b} - \int_1^b \frac{\cos x}{x^2}\,dx.$$

Show that the improper integral $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ exists. See Question 10.

Solution:

$$\begin{aligned}
\int_1^b \frac{\sin x}{x}\,dx = \int_1^b \frac{1}{x}\frac{d}{dx}\big(-\cos x\big)\,dx &= \Big[-\frac{\cos x}{x}\Big]_1^b - \int_1^b \big(-\cos x\big)\Big(-\frac{1}{x^2}\Big)\,dx \\
&= \cos(1) - \frac{\cos b}{b} - \int_1^b \frac{\cos x}{x^2}\,dx.
\end{aligned}$$

Now use the comparison $\left|\frac{\cos x}{x^2}\right| \leq \frac{1}{x^2}$ and the fact that $\int_1^\infty \frac{1}{x^2}\,dx$ converges. The Comparison Test shows that $\int_1^\infty \frac{\cos x}{x^2}\,dx$ exists. That is, $\lim_{b\to\infty}\int_1^b \frac{\cos x}{x^2}\,dx$ exists, and equals L, say. Clearly $(\cos b)/b \to 0$ as $b \to \infty$. Hence,

$$\int_1^b \frac{\sin x}{x}\,dx = \cos(1) - \frac{\cos b}{b} - \int_1^b \frac{\cos x}{x^2}\,dx \to \cos(1) - L$$

as $b \to \infty$. Thus $\int_1^\infty \frac{\sin x}{x}\, dx$ exists. Since $\frac{\sin x}{x}$ has a *removable singularity* at $x = 0$ the integral $\int_0^1 \frac{\sin x}{x}\, dx$ exists (think instead of integrating the continuous function $f(x) = \frac{\sin x}{x}$ if $x \neq 0$ and $f(0) = 1$). Therefore

$$\int_0^\infty \frac{\sin x}{x}\, dx = \int_0^1 \frac{\sin x}{x}\, dx + \int_1^\infty \frac{\sin x}{x}\, dx$$

exists

The point of using integration by parts is that the Comparison Test is not immediately applicable to the function $(\sin x)/x$ on $[1, \infty)$.

6. Make the substitution $x = \pi - t$ to show that if f is continuous then

$$\int_0^\pi x f(\sin x)\, dx \;=\; \frac{\pi}{2} \int_0^\pi f(\sin x)\, dx.$$

Solution: According to the Change of Variable Rule for definite integrals,

$$\begin{aligned} I \;=\; \int_0^\pi x f(\sin x)\, dx \;&=\; \int_\pi^0 (\pi - t) f\big(\sin(\pi - t)\big)(-1)\, dt \\ &= \int_0^\pi (\pi - t) f\big(\sin(\pi - t)\big)\, dt \\ &= \int_0^\pi (\pi - t) f(\sin t)\, dt \\ &= \pi \int_0^\pi f(\sin t)\, dt \;-\; \int_0^\pi t f(\sin t)\, dt \\ &= \pi \int_0^\pi f(\sin t)\, dt \;-\; \int_0^\pi x f(\sin x)\, dx \\ &= \pi \int_0^\pi f(\sin t)\, dt \;-\; I. \end{aligned}$$

Now add I to both sides and divide by 2, and we get the stated formula.

Extra questions

7. Find a reduction formula for the integral $I_n = \dfrac{1}{n!}\displaystyle\int_0^1 (1-x^2)^n \cos\left(\tfrac12\pi x\right)dx.$

Solution: Let $u = (1-x^2)^n$ and $\frac{dv}{dx} = \cos\left(\tfrac12\pi x\right)$. Then for $n \geq 1$,

$$\begin{aligned}I_n &= \frac{1}{n!}\left\{\left[\frac{2}{\pi}(1-x^2)^n\sin\left(\tfrac12\pi x\right)\right]_0^1 + \frac{4n}{\pi}\int_0^1 x(1-x^2)^{n-1}\sin\left(\tfrac12\pi x\right)dx\right\}\\ &= \frac{4}{\pi(n-1)!}\int_0^1 x(1-x^2)^{n-1}\sin\left(\tfrac12\pi x\right)dx.\end{aligned}$$

Now let $u = x(1-x^2)^{n-1}$ and $\frac{dv}{dx} = \sin\left(\tfrac12\pi x\right)$. Then

$$\frac{du}{dx} = (1-x^2)^{n-1} - 2(n-1)x^2(1-x^2)^{n-2} = (2n-1)(1-x^2)^{n-1} - 2(n-1)(1-x^2)^{n-2}$$

and $v = -(2/\pi)\cos\left(\tfrac12\pi x\right)$. Thus for $n \geq 2$,

$$\begin{aligned}I_n &= \frac{4}{\pi(n-1)!}\left\{\left[-\frac{2}{\pi}x(1-x^2)^{n-1}\cos\left(\tfrac12\pi x\right)\right]_0^1\right.\\ &\qquad + \frac{2(2n-1)}{\pi}\int_0^1 (1-x^2)^{n-1}\cos\left(\tfrac12\pi x\right)dx\\ &\qquad \left. - \frac{4(n-1)}{\pi}\int_0^1 (1-x^2)^{n-2}\cos\left(\tfrac12\pi x\right)dx\right\}\\ &= \frac{4}{\pi(n-1)!}\left\{\frac{2(2n-1)}{\pi}(n-1)!I_{n-1} - \frac{4(n-1)}{\pi}(n-2)!I_{n-2}\right\}\\ &= \frac{8(2n-1)}{\pi^2}I_{n-1} - \frac{16}{\pi^2}I_{n-2}\,.\end{aligned}$$

Hence the reduction formula is

$$I_n = \frac{8}{\pi^2}\Big((2n-1)I_{n-1} - 2I_{n-2}\Big) \qquad (n \geq 2).$$

Remark: This reduction formula can be used to give a quick proof of the irrationality of π and π^2. First, calculate $I_0 = 2/\pi$ and $I_1 = 16/\pi^3$. Then, repeated application of the reduction formula shows that I_n is a polynomial in $1/\pi$ of degree $2n+1$ having integer coefficients. In addition, the polynomial is odd (i.e., only odd powers of $1/\pi$ appear). Now suppose that π^2 is rational. This means that we can write $\pi^2 = p/q$, where p and q are (positive) integers with no common factor. When this is substituted into the expression for πI_n, it is seen that πI_n becomes a rational fraction with denominator p^n, after which it may be possible to cancel the fraction down to one with smaller denominator. In any case, $\pi p^n I_n$ is an integer, say, C_n, for all positive integers n. But, from the definition of I_n, we have the inequality $0 < I_n < 1/n!$. Hence, $0 < C_n < \pi p^n/n!$. Because factorials grow more rapidly than powers, we can always choose n sufficiently large so that $0 < C_n < \frac12$, giving a contradiction.

8. Let $x \in \mathbb{R}$, and let $n \geq 0$ be an integer.

(a) Use a reduction formula to prove that

$$\int_0^x (x-t)^n e^t \, dt = n! \left(e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \right).$$

Solution: Let $u = (x-t)^n$ and $\frac{dv}{dx} = e^t$. Then

$$I_n = \int_0^x (x-t)^n e^t \, dt = -x^n + nI_{n-1}.$$

The stated formula now follows by induction (using the reduction formula for the induction step).

(b) Set $x = 1$ in (a) and deduce that e is irrational.

Hint: If e is rational then the integral is an integer for sufficiently large n.

Solution: Suppose that $e = \frac{a}{b}$ is rational. Then

$$n! \left(\frac{a}{b} - \sum_{k=0}^{n} \frac{1}{k!} \right) = \int_0^1 (1-t)^n e^t \, dt.$$

The left hand side is an integer whenever $n \geq b$, and therefore the integral is an integer for all sufficiently large n. But

$$0 < \int_0^1 (1-t)^n e^t \, dt \leq e \int_0^1 (1-t)^n \, dt < \frac{3}{n+1},$$

giving a contradiction.

(c) Use (a) to show that

$$\lim_{n \to \infty} \sum_{k=0}^{n} \frac{x^k}{k!} = e^x \qquad \text{for all } x \in \mathbb{R}.$$

Solution: If $x \geq 0$ then

$$0 \leq \int_0^x (x-t)^n e^t \, dt \leq e^x \int_0^x (x-t)^n \, dt = \frac{x^{n+1} e^x}{n+1}.$$

If $x < 0$ then

$$\left| \int_0^x (x-t)^n e^t \, dt \right| = \left| -(-1)^n \int_x^0 (t-x)^n e^t \, dt \right| = \int_x^0 (t-x)^n e^t \, dt.$$

Therefore in this case,

$$0 \leq \left| \int_0^x (x-t)^n e^t \, dt \right| \leq \int_x^0 (t-x)^n \, dt = \frac{(-x)^{n+1}}{n+1} = \frac{|x|^{n+1}}{n+1}.$$

Therefore for all $x \in \mathbb{R}$ we have

$$0 \leq \left| e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \right| \leq \frac{|x|^{n+1}}{(n+1)!} \max\{1, e^x\}.$$

For each *fixed* x the right hand side tends to 0 as $n \to \infty$, and so by the squeeze law

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!} = e^x \qquad \text{for all } x \in \mathbb{R}.$$

Remark: In this question you actually showed that the Taylor series for e^x converges to e^x for all $x \in \mathbb{R}$. We will be discussing Taylor series later in the course, and will reprove this result using different methods.

9. For $n \geq 0$ let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$.

(a) Derive a reduction formula for I_n, and use it to deduce that

$$I_{2n} = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2} \qquad \text{and} \qquad I_{2n+1} = \frac{(2n)!!}{(2n+1)!!},$$

where $(2n)!! = 2 \cdot 4 \cdots (2n)$ and $(2n+1)!! = 1 \cdot 3 \cdots (2n+1)$.

Solution: We have

$$I_n = \int_0^{\frac{\pi}{2}} (1-\cos^2\theta)\sin^{n-2}\theta \, d\theta = I_{n-2} - \int_0^{\frac{\pi}{2}} \cos^2\theta \sin^{n-2}\theta \, d\theta.$$

Using integration by parts (with $u = \cos\theta$ and $\frac{dv}{d\theta} = \cos\theta\sin^{n-2}\theta$) gives

$$I_n = I_{n-2} - \frac{1}{n-1}\int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta = I_{n-2} - \frac{1}{n-1} I_n.$$

Solving for I_n gives the reduction formula

$$I_n = \frac{n-1}{n} I_{n-2} \quad \text{for } n \geq 1.$$

Therefore

$$I_{2n} = \frac{(2n-1)(2n-3)\cdots 3 \cdot 1}{(2n)(2n-2)\cdots 4 \cdot 2} I_0 = \frac{(2n-1)!!}{(2n)!!}\int_0^{\frac{\pi}{2}} 1 \, d\theta = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}$$

and

$$I_{2n+1} = \frac{(2n)(2n-2)\cdots 4 \cdot 2}{(2n+1)(2n-1)\cdots 5 \cdot 3} I_1 = \frac{(2n)!!}{(2n+1)!!}\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta = \frac{(2n)!!}{(2n+1)!!}.$$

(b) Show that $I_{2n-1} \leq I_{2n} \leq I_{2n+1}$, and deduce that that

$$1 \leq \frac{1 \cdot 3 \cdot 3 \cdots (2n-1)(2n-1)}{2 \cdot 2 \cdot 4 \cdots (2n-2)(2n)}\frac{\pi}{2} \leq \frac{2n}{2n+1}.$$

Hence prove the *Wallis Product Formula* for π:

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7}\cdots = \lim_{n\to\infty} \frac{2^{4n} n!^4}{2n(2n)!^2}.$$

Solution: Since $0 \leq \sin^{2n-1}\theta \leq \sin^{2n}\theta \leq \sin^{2n+1}\theta$ for all $0 \leq \theta \leq \frac{\pi}{2}$ we have $I_{2n-1} \leq I_{2n} \leq I_{2n+1}$. Therefore by part (a) we have

$$\frac{(2n-2)!!}{(2n-1)!!} \leq \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2} \leq \frac{(2n)!!}{(2n+1)!!}.$$

Rearranging gives

$$1 \leq \frac{1 \cdot 3 \cdot 3 \cdots (2n-1)(2n-1)}{2 \cdot 2 \cdot 4 \cdots (2n-2)(2n)} \frac{\pi}{2} \leq \frac{2n}{2n+1},$$

and it follows from the Squeeze Theorem that

$$\frac{\pi}{2} = \lim_{n\to\infty} \left(\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n}{2n-1} \right).$$

Since

$$(2n)!! = 2^n n! \quad \text{and} \quad (2n-1)!! = \frac{(2n)!}{(2n)!!} = \frac{(2n)!}{2^n n!}$$

we can rewrite the above limit as

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{(2n)!!^2}{2n(2n-1)!!^2} = \lim_{n\to\infty} \frac{2^{4n} n!^4}{2n(2n)!^2}.$$

10. For integers $n \geq 1$ let

$$a_n = \int_0^{\frac{\pi}{2}} \sin(2nx) \cot x \, dx \qquad \text{and} \qquad b_n = \int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{x} \, dx.$$

(a) Prove the formula

$$\frac{\sin(2nx)}{\sin x} = 2 \sum_{k=1}^{n} \cos\left((2k-1)x\right) \qquad \text{for } x \notin \pi\mathbb{Z} \text{ and } n \geq 1.$$

Solution: By induction. For $n = 1$ the result is true by the double angle formula $\sin(2x) = 2 \sin x \cos x$. For the induction step, note that

$$\begin{aligned}
\frac{\sin\left(2(n+1)x\right)}{\sin x} &= \frac{\sin(2nx + 2x)}{\sin x} \\
&= \frac{\sin(2nx)\cos(2x) + \cos(2nx)\sin(2x)}{\sin x} \\
&= \frac{\sin(2nx)(\cos^2 x - \sin^2 x) + 2\cos(2nx)\sin x \cos x}{\sin x} \\
&= \frac{\sin(2nx)(1 - 2\sin^2 x)}{\sin x} + 2\cos(2nx)\cos x \\
&= \frac{\sin(2nx)}{\sin x} + 2\left(\cos(2nx)\cos x - \sin(2nx)\sin x\right) \\
&= \frac{\sin(2nx)}{\sin x} + 2\cos\left((2n+1)x\right).
\end{aligned}$$

Applying the induction hypothesis completes the proof.

(b) Deduce that $a_n = \frac{\pi}{2}$ for all $n \geq 1$.

Solution: Using (a) we have

$$a_n = \int_0^{\frac{\pi}{2}} \frac{\sin(2nx)}{\sin x} \cos x \, dx = 2 \sum_{k=1}^{n} \int_0^{\frac{\pi}{2}} \cos\left((2k-1)x\right) \cos x \, dx.$$

The formula $\cos A \cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$ implies that

$$a_n = \sum_{k=1}^{n} \int_0^{\frac{\pi}{2}} \Big(\cos(2kx) + \cos\left((2k-2)x\right)\Big) \, dx.$$

If $k \geq 1$ then

$$\int_0^{\frac{\pi}{2}} \cos(2kx) \, dx = \frac{1}{2k} \sin(2kx)\Big|_0^{\frac{\pi}{2}} = 0.$$

Hence the only surviving term comes from $\cos((2k-2)x)$ with $k = 1$, and so

$$a_n = \int_0^{\frac{\pi}{2}} 1 \, dx = \frac{\pi}{2}.$$

(c) Use integration by parts to show that $\lim_{n\to\infty} (a_n - b_n) = 0$.

Solution: This part is pretty tricky! First we write

$$a_n - b_n = \lim_{\epsilon\to 0} \int_\epsilon^{\frac{\pi}{2}} \sin(2nx) \left(\cot x - x^{-1}\right) \, dx.$$

Let $u = \cot x - x^{-1}$ and $\frac{dv}{dx} = \sin(2nx)$. Then

$$\frac{du}{dx} = x^{-2} - \operatorname{cosec}^2 x \qquad \text{and} \qquad v = -\frac{1}{2n}\cos(2nx),$$

and so by integration by parts we have

$$\int_\epsilon^{\frac{\pi}{2}} \sin(2nx)\left(\cot x - x^{-1}\right) dx = \frac{\cos n\pi}{n\pi} + \frac{\cos(2n\epsilon)}{2n}\left(\cot \epsilon - \epsilon^{-1}\right) + \frac{1}{2n}\int_\epsilon^{\frac{\pi}{2}} \cos(2nx)\left(x^{-2} - \operatorname{cosec}^2 x\right) dx.$$

A straightforward (but tedious) application of L'Hôpital's Rule shows that

$$\lim_{x\to 0}\left(x^{-2} - \operatorname{cosec}^2 x\right) = -\frac{1}{3}$$

Therefore the function $\cos(2nx)\left(x^{-2} - \operatorname{cosec}^2 x\right)$ has a removable discontinuity at $x = 0$, and so

$$\lim_{\epsilon\to 0}\int_\epsilon^{\frac{\pi}{2}} \cos(2nx)\left(x^{-2} - \operatorname{cosec}^2 x\right) dx = \int_0^{\frac{\pi}{2}} \cos(2nx)\left(x^{-2} - \operatorname{cosec}^2 x\right) dx$$

exists.

Another application of L'Hôpital's Rule gives

$$\lim_{\epsilon\to 0}\left(\cot\epsilon - \epsilon^{-1}\right) = 0,$$

and therefore

$$\begin{aligned}
a_n - b_n &= \lim_{\epsilon\to 0}\int_{\epsilon}^{\frac{\pi}{2}} \sin(2nx)\left(\cot x - x^{-1}\right)\,dx \\
&= \frac{\cos n\pi}{n\pi} + \frac{1}{2n}\lim_{\epsilon\to 0}\Bigg(\cos(2n\epsilon)\left(\cot\epsilon - \epsilon^{-1}\right)\Bigg) \\
&\quad + \frac{1}{2n}\lim_{\epsilon\to 0}\int_{\epsilon}^{\frac{\pi}{2}}\cos(2nx)\left(x^{-2} - \operatorname{cosec}^2 x\right)\,dx \\
&= \frac{\cos n\pi}{n\pi} + \frac{1}{2n}\int_{0}^{\frac{\pi}{2}}\cos(2nx)\left(x^{-2} - \operatorname{cosec}^2 x\right)\,dx.
\end{aligned}$$

Now take $n\to\infty$. We have

$$\lim_{n\to\infty}\frac{\cos n\pi}{n\pi} = 0,$$

and since

$$0 \le \left|\frac{1}{2n}\int_0^{\frac{\pi}{2}}\cos(2nx)\left(x^{-2} - \operatorname{cosec}^2 x\right)\,dx\right| \le \frac{1}{2n}\int_0^{\frac{\pi}{2}}\left|x^{-2} - \operatorname{cosec}^2 x\right|dx$$

the Squeeze Law gives

$$\lim_{n\to\infty}\frac{1}{2n}\int_0^{\frac{\pi}{2}}\cos(2nx)\left(x^{-2} - \operatorname{cosec}^2 x\right)\,dx = 0.$$

Thus

$$\lim_{n\to\infty}(a_n - b_n) = 0 + 0 = 0.$$

(d) Deduce that $\displaystyle\int_0^{\infty}\frac{\sin x}{x}\,dx = \frac{\pi}{2}.$

Solution: Parts (b) and (c) give

$$\lim_{n\to\infty} b_n = \frac{\pi}{2}.$$

Thus by making by the change of variable $u = 2nx$ we have

$$\frac{\pi}{2} = \lim_{n\to\infty}\int_0^{\frac{\pi}{2}}\frac{\sin(2nx)}{x}\,dx = \lim_{n\to\infty}\int_0^{n\pi}\frac{\sin u}{u}\,du = \int_0^{\infty}\frac{\sin u}{u}\,du.$$