The University of Sydney
School of Mathematics and Statistics

Tutorial for Week 6

MATH1903: Integral Calculus and Modelling (Advanced) — Semester 2, 2009

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Questions to attempt in class

1. Decide if the following sequences converge. If they converge find the limit.

(a) $a_n = \dfrac{3 + \cos n^2}{\sqrt{n}}$

(b) $a_n = \sqrt[n]{n}$

(c) $a_n = \dfrac{n^2}{3n^2 + 2n - 1}$

(d) $a_n = \dbinom{2n}{n}$

2. Decide if the following series converge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2 - \sin\sqrt{n}}{n^3}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - 2n + 5}{n^3 + 4}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n!}$

(d) $\displaystyle\sum_{n=1}^{\infty} \sin(n^2)$

3. (a) Show that the improper integral $\displaystyle\int_2^{\infty} \frac{dx}{x \ln x}$ diverges to ∞.

(b) Deduce that the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ diverges to ∞. *Hint: Use Riemann sums.*

(c) The Prime Number Theorem implies that the nth prime satisfies $p_n \sim n \ln n$. Given this information, show that the series

$$\sum_{\text{primes } p} \frac{1}{p} \qquad \text{diverges to } \infty.$$

Questions for extra practice

4. Decide if the following sequences converge. If they converge find the limit.

(a) $a_n = \dfrac{1 + 2 + \cdots + n}{n^2}$

(b) $a_n = e^{-n} \cosh n$

(c) $a_n = \left(1 + \dfrac{1}{n}\right)^n$

(d) $a_n = \dfrac{n^n}{n!}$

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5. Decide if the following series converge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cosh n}{n^4+1}$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^k}$

(b) $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n}$

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{\ln k}}$

6. Show that the series

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} \frac{x^{2k+1}}{2k+1}$$

converges if $|x| < 1$, and diverges if $|x| > 1$. *Hint: Use the ratio test.*

7. Use Riemann sums to show that

$$\ln n + \frac{1}{n} \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq \ln n + 1,$$

and deduce that $1 + \frac{1}{2} + \cdots + \frac{1}{n} \sim \ln n$.

8. Let (a_n) and (b_n) be sequences with $a_n, b_n \to \infty$ and $a_n \sim b_n$.

(a) Show that $\ln a_n \sim \ln b_n$.

(b) Is it necessarily true that $e^{a_n} \sim e^{b_n}$?

9. Write down a proof of the Squeeze Law for sequences: If $a_n \leq b_n \leq c_n$ for all large n, and if $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = \ell$, then $\lim\limits_{n\to\infty} b_n = \ell$ too.

Stirling's Formula (for interest)

10. In this question you derive Stirling's Asymptotic Formula for $n!$

(a) Show that

$$\ln n! = n \ln n - n + 1 + \int_1^n \frac{\{x\}}{x} dx\,,$$

where $\{x\} \in [0, 1)$ is the fractional part of $x \geq 0$.

Hint: Notice that $\displaystyle\int_1^n \frac{\{x\}}{x}\, dx = \sum_{k=1}^{n-1} \int_k^{k+1} \frac{x-k}{x}\, dx.$

(b) Integrate by parts to show that

$$\int_1^n \frac{\{x\}}{x}\, dx = \frac{1}{2} \ln n - \frac{1}{2} \int_1^n \frac{\{x\} - \{x\}^2}{x^2}\, dx.$$

(c) Deduce that $\displaystyle\lim_{n\to\infty} \frac{n!}{\sqrt{n}n^n e^{-n}} = e^C$ for some constant C.

(d) Use the Wallis product formula from last week to evaluate C, and deduce that

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n}.$$