The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial for Week 6

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2009

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Questions to attempt in class

1. Decide if the following sequences converge. If they converge find the limit.

(a) $a_n = \dfrac{3+\cos n^2}{\sqrt{n}}$

Solution: Since
$$0 \le \left|\frac{3+\cos n^2}{\sqrt{n}}\right| \le \frac{4}{\sqrt{n}}$$
we have $a_n \to 0$ by the Squeeze Law.

(b) $a_n = \sqrt[n]{n}$

Solution: We have
$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{x}{\ln x}} = e^{\lim_{x\to\infty} \frac{x}{\ln x}}.$$
By L'Hôpital's Rule $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$, and so $a_n \to e^0 = 1$.

(c) $a_n = \dfrac{n^2}{3n^2+2n-1}$

Solution: Dividing the numerator and denominator by n^2 shows that
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3+\frac{2}{n}-\frac{1}{n}} = \frac{1}{3+0-0} = \frac{1}{3}.$$

(d) $a_n = \dbinom{2n}{n}$

Solution: Since $\dbinom{2n}{n} = \dfrac{(2n)!}{n!^2}$ it seems difficult to compute the limit of a_n directly. Factorials are often best treated using the ratio test for sequences.
$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(2n+2)!n!^2}{(n+1)!^2(2n)!} = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)^2} = 4 > 1.$$
Therefore by the ratio test $|a_n|$ diverges to ∞, and therefore $a_n \to \infty$.

2. Decide if the following series converge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2-\sin\sqrt{n}}{n^3}$

Solution: Since

$$\left|\frac{2-\sin\sqrt{n}}{n^3}\right| \leq \frac{3}{n^3},$$

and since $\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges, we deduce that the original series converges too by the Comparison Test.

(b) $\sum_{n=1}^{\infty}\frac{n^2-2n+5}{n^3+4}$

Solution: Since

$$\frac{n^2-2n+5}{n^3+4} \sim \frac{1}{n},$$

and since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, we see that the original series diverges too by the asymptotic comparison test.

(c) $\sum_{n=1}^{\infty}\frac{5^n}{n!}$

Solution: Again, factorials are often best treated using the ration test: Let $a_n = \frac{5^n}{n!}$. Then

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{5}{n+1} = 0 < 1,$$

and so by the ratio test for series we see that the series converges.

(d) $\sum_{n=1}^{\infty}\sin(n^2)$

Solution: We say in class that if a series converges then $a_n \to 0$. Therefore this series cannot converge.

3. (a) Show that the improper integral $\int_2^{\infty}\frac{dx}{x\ln x}$ diverges to ∞.

Solution: After making the change of variable $u = \ln x$ we have

$$\int_2^b \frac{dx}{x\ln x} = \int_{\ln 2}^{\ln b} u^{-1}\,du = \ln(\ln b) - \ln(\ln 2).$$

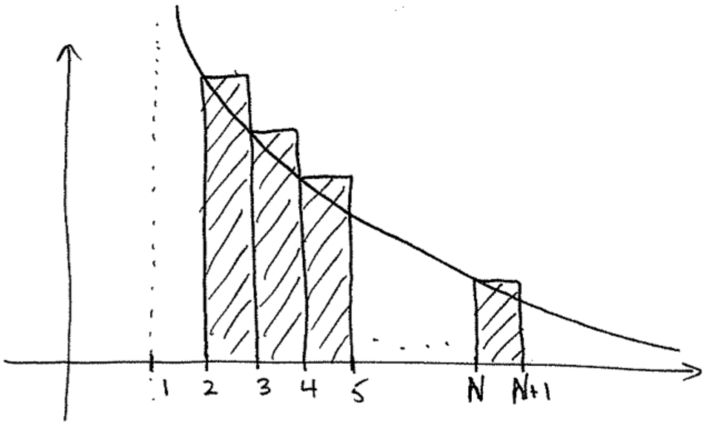
Therefore

$$\lim_{b\to\infty}\int_2^b \frac{dx}{x\ln x} = \infty,$$

and so the improper integral diverges to ∞.

(b) Deduce that the series $\sum_{k=2}^{\infty}\frac{1}{k\ln k}$ diverges to ∞. *Hint: Use Riemann sums.*

Solution: Some simple calculus shows that $f(x) = \dfrac{1}{x\ln x}$ is monotonically decreasing on $(1,\infty)$. Therefore it looks something like the picture below.



Using the upper Riemann sum as shown,

$$\sum_{k=2}^{N} \frac{1}{k \ln k} \geq \int_{2}^{N+1} \frac{dx}{x \ln x}.$$

By the previous part the integral diverges to ∞ as $N \to \infty$, and therefore the series diverges to ∞ too.

(c) The Prime Number Theorem implies that the nth prime satisfies $p_n \sim n \ln n$. Given this information, show that the series

$$\sum_{\text{primes } p} \frac{1}{p} \qquad \text{diverges to } \infty.$$

Solution: The nth term of the series is $a_n = \frac{1}{p_n}$, where p_n is the nth prime. We are told that $p_n \sim n \ln n$, and so $a_n \sim \frac{1}{n \ln n}$. Therefore the series diverges to ∞ (using the Asymptotic Comparison Test and the previous part).

Questions for extra practice

4. Decide if the following sequences converge. If they converge find the limit.

(a) $a_n = \dfrac{1 + 2 + \cdots + n}{n^2}$

Solution: Since $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$ we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n(n+1)}{2n^2} = \lim_{n\to\infty} \frac{1 + \frac{1}{n}}{2} = \frac{1}{2}.$$

(b) $a_n = e^{-n} \cosh n$

Solution: Using the definition of cosh we have $a_n = \frac{1}{2}(1 + e^{-2n})$. Therefore $a_n \to \frac{1}{2}$ as $n \to \infty$.

(c) $a_n = \left(1 + \dfrac{1}{n}\right)^n$

Solution: We have

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x \ln\left(1+\frac{1}{x}\right)} = e^{\lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right)}.$$

By L'Hôpital's Rule,

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1.$$

Therefore $a_n \to e^1 = e$.

(d) $a_n = \dfrac{n^n}{n!}$

Solution: You can see using the ratio test that this sequence diverges (you'll need to use part (a)). More simply:

$$a_n = \frac{n}{n}\times\frac{n}{n-1}\times\cdots\times\frac{n}{1} \geq \underbrace{1\times 1\times\cdots\times 1}_{n-1 \text{ factors}}\times n = n,$$

which shows that the sequence diverges to ∞.

5. Decide if the following series converge.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{\cosh n}{n^4+1}$

Solution: Using the definition of cosh we see that $a_n \to \infty$ as $n \to \infty$. Therefore this series does not converge.

(b) $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n}$

Solution: We have

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 e^{-1} = e^{-1} < 1,$$

and so the series converges by the ratio test.

(c) $\displaystyle\sum_{k=1}^{\infty}\frac{k!}{k^k}$

Solution: You may use the ratio test to see that the series converges:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to e^{-1},$$

where we have used Question 4(d). Alternatively,

$$a_n = \frac{n}{n}\times\frac{n-1}{n}\times\cdots\times\frac{3}{n}\times\frac{2}{n}\times\frac{1}{n} \leq 1\times\cdots\times 1\times\frac{2}{n}\times\frac{1}{n} = \frac{2}{n^2}.$$

Therefore the series converges by comparison with $\sum\frac{1}{k^2}$.

(d) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^{\ln k}}$

Solution: Once $k \geq e^2$ we have $\ln k \geq 2$, and so

$$\frac{1}{k^{\ln k}} \leq \frac{1}{k^2}.$$

Therefore the series converges by comparison to $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$.

6. Show that the series

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} \frac{x^{2k+1}}{2k+1}$$

converges if $|x| < 1$, and diverges if $|x| > 1$. *Hint: Use the ratio test.*

Solution: Let $a_n = \frac{\binom{2n}{n}}{2^{2n}} \frac{x^{2n+1}}{2n+1} = \frac{(2n)!}{2^{2n} n!^2 (2n+1)} x^{2n+1}$. Then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(2n+1)(2n+2)(2n+1)}{2^2(n+1)^2(2n+3)} |x|^2 = |x|^2.$$

Therefore by the ratio test the series converges if $|x| < 1$, and diverges if $|x| > 1$.

In fact this series converges when $|x| = 1$ too. To see this, apply Stirling's formula (Question 10) to see that

$$\frac{\binom{2n}{n}}{2^{2n}(2n+1)} \sim \frac{1}{2\sqrt{\pi}\, n^{3/2}}.$$

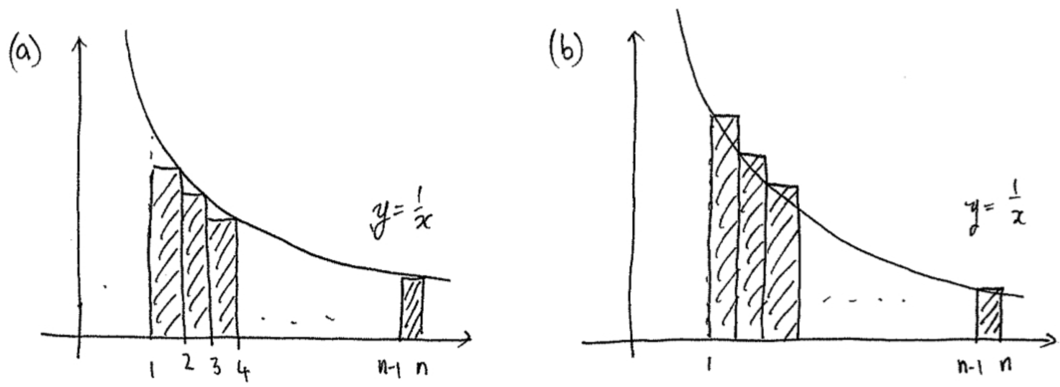
Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, so does the original series by the asymptotic version of the Comparison Test.

7. Use Riemann sums to show that

$$\ln n + \frac{1}{n} \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq \ln n + 1,$$

and deduce that $1 + \frac{1}{2} + \cdots + \frac{1}{n} \sim \ln n$.

Solution: Consider the following pictures:



Picture (a) shows that

$$\frac{1}{2} + \cdots + \frac{1}{n} \leq \int_1^n \frac{1}{x}\, dx = \ln n,$$

and so $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \le 1+\ln n$. Picture (b) shows that

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1} \ge \int_1^n \frac{1}{x}\,dx = \ln n,$$

and so $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \ge \ln n + \frac{1}{n}$.

Then

$$1+\frac{1}{n\ln n} \le \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{\ln n} \le 1+\frac{1}{\ln n}.$$

The terms on the left and right tend to 1 as $n\to\infty$, and therefore by the Squeeze Law we have

$$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{\ln n} = 1.$$

That is, $1+\frac{1}{2}+\cdots+\frac{1}{n} \sim \ln n$.

8. Let (a_n) and (b_n) be sequences with $a_n, b_n \to \infty$ and $a_n \sim b_n$.

(a) Show that $\ln a_n \sim \ln b_n$.

Solution: Write

$$\frac{\ln a_n}{\ln b_n} = \frac{\ln a_n - \ln b_n}{\ln b_n} + 1.$$

But $\frac{\ln a_n - \ln b_n}{\ln b_n} = \frac{\ln(a_n/b_n)}{\ln b_n}$. Since $a_n \sim b_n$ we have $a_n/b_n \to 1$ as $n\to\infty$, and therefore $\ln(a_n/b_n) \to 0$ as $n\to\infty$. Also, since $b_n\to\infty$ we have $\ln b_n \to \infty$, and therefore

$$\lim_{n\to\infty} \frac{\ln a_n - \ln b_n}{\ln b_n} = 0.$$

Thus $\lim_{n\to\infty} \frac{\ln a_n}{\ln b_n} = 1$, which is the definition of $\ln a_n \sim \ln b_n$.

(b) Is it necessarily true that $e^{a_n} \sim e^{b_n}$?

Solution: No, this is not necessarily true. Consider $a_n = n^2+n$ and $b_n = n^2$. Then $a_n/b_n \to 1$, and so $a_n \sim b_n$. But

$$\lim_{n\to\infty} \frac{e^{a_n}}{e^{b_n}} = \lim_{n\to\infty} e^n \to \infty,$$

and so e^{a_n} is not asymptotic to e^{b_n}.

9. Write down a proof of the Squeeze Law for sequences: If $a_n \le b_n \le c_n$ for all large n, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \ell$, then $\lim_{n\to\infty} b_n = \ell$ too.

Solution: Suppose that $a_n \le b_n \le c_n$ for all $n \ge N_0$. Then

$$a_n - \ell \le b_n - \ell \le c_n - \ell \qquad \text{for all } n \ge N_0.$$

Let $\epsilon > 0$ be given.

- Since $a_n \to \ell$, there is $N_1 \in \mathbb{N}$ such that

$$-\epsilon < a_n - \ell < \epsilon \qquad \text{whenever } n \geq N_1.$$

- Since $c_n \to \ell$, there is $N_2 \in \mathbb{N}$ such that

$$-\epsilon < c_n - \ell < \epsilon \qquad \text{whenever } n \geq N_2.$$

Therefore if $n \geq N = \max\{N_0, N_1, N_2\}$ then

$$-\epsilon < a_n - \ell \leq b_n - \ell \leq c_n - \ell < \epsilon.$$

Therefore

$$|b_n - \ell| < \epsilon \qquad \text{whenever } n \geq N,$$

and so by the definition of limits $b_n \to \ell$ as $n \to \infty$.

Stirling's Formula (for interest)

10. In this question you derive Stirling's Asymptotic Formula for $n!$

(a) Show that

$$\ln n! = n \ln n - n + 1 + \int_1^n \frac{\{x\}}{x} dx\,,$$

where $\{x\} \in [0, 1)$ is the fractional part of $x \geq 0$.

Hint: Notice that $\displaystyle\int_1^n \frac{\{x\}}{x}\, dx = \sum_{k=1}^{n-1} \int_k^{k+1} \frac{x-k}{x}\, dx.$

Solution: Using the hint, we have

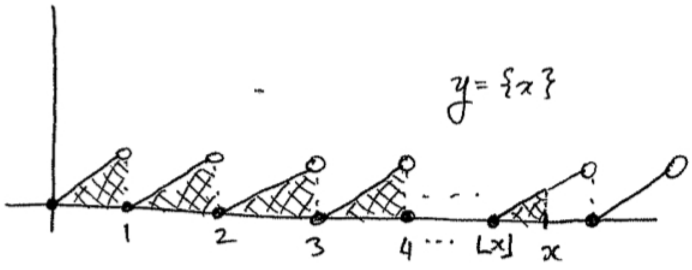
$$\begin{aligned}
\int_1^n \frac{\{x\}}{x}\, dx &= \sum_{k=1}^{n-1} \int_k^{k+1} \frac{x-k}{x}\, dx \\
&= \sum_{k=1}^{n-1} (x - k \ln x) \Big|_k^{k+1} \\
&= \sum_{k=1}^{n-1} (1 - k \ln(k+1) + k \ln k) \\
&= n - 1 - (\ln 2 + 2 \ln 3 + 3 \ln 4 + \cdots + (n-1) \ln n) \\
&\qquad + (\ln 1 + 2 \ln 2 + 3 \ln 3 + \cdots + (n-1) \ln(n-1)) \\
&= n - 1 - n \ln n + (\ln 1 + \ln 2 + \ln 3 + \cdots + \ln n) \\
&= n - 1 - n \ln n + \ln n!.
\end{aligned}$$

Rearranging gives the desired result.

(b) Integrate by parts to show that

$$\int_1^n \frac{\{x\}}{x}\, dx = \frac{1}{2} \ln n - \frac{1}{2} \int_1^n \frac{\{x\} - \{x\}^2}{x^2}\, dx.$$

Solution: In the integration by parts formula, let $u = \frac{1}{x}$ and $\frac{dv}{dx} = \{x\}$. Then $\frac{du}{dx} = -\frac{1}{x^2}$. We compute $v = \displaystyle\int \{x\}\, dx$ by an area computation:



The area under the graph from 0 to x is

$$v = \frac{1}{2}\lfloor x \rfloor + \frac{1}{2}\{x\}^2 = \frac{1}{2}\left(x - \{x\} + \{x\}^2\right),$$

where $\lfloor x \rfloor \in \mathbb{Z}$ is the *integer part* of x.
Therefore

$$\begin{aligned} \int_1^n \frac{\{x\}}{x}\, dx &= \left.\frac{x - \{x\} + \{x\}^2}{2x}\right|_1^n + \frac{1}{2}\int_1^n \frac{x - \{x\} + \{x\}^2}{x^2}\, dx \\ &= \frac{1}{2}\ln n - \frac{1}{2}\int_1^n \frac{\{x\} - \{x\}^2}{x^2}\, dx. \end{aligned}$$

(c) Deduce that $\lim_{n\to\infty} \frac{n!}{\sqrt{n}n^n e^{-n}} = e^C$ for some constant C.

Solution: Let $C_n = 1 - \frac{1}{2}\int_1^n \frac{\{x\} - \{x\}^2}{x^2}\, dx$. Since

$$\left|\frac{\{x\} - \{x\}^2}{x^2}\right| \le \frac{2}{x^2}$$

we see that $\lim_{n\to\infty} C_n = C$ exists by Comparison to $\int_1^\infty \frac{1}{x^2}\, dx$.
By the previous parts we have

$$\ln n! = n\ln n - n + \frac{1}{2}\ln n + C_n,$$

and therefore $n! = n^n e^{-n}\sqrt{n}e^{C_n}$, and so

$$\frac{n!}{\sqrt{n}n^n e^{-n}} = e^{C_n} \to e^C \qquad \text{as } n \to \infty.$$

(d) Use the Wallis product formula from last week to evaluate C, and deduce that

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n}.$$

Solution: The Wallis Formula from last week gives

$$\pi = \lim_{n\to\infty} \frac{2^{4n} n!^4}{n(2n)!^2}.$$

Plugging $n! = \sqrt{n}n^n e^{-1} e^{C_n}$ into this formula gives

$$\pi = \lim_{n\to\infty} \frac{2^{4n} n^2 n^{4n} e^{-4n} e^{4C_n}}{n(2n)(2n)^{4n} e^{-4n} e^{2C_{2n}}} = \frac{1}{2} \lim_{n\to\infty} e^{4C_n - 2C_{2n}} = \frac{1}{2} e^{2C}.$$

Therefore $C = \dfrac{1}{2}\ln(2\pi)$. Therefore by (c) we have

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\, n^n e^{-n}} = 1, \qquad \text{and so} \qquad n! \sim \sqrt{2\pi n}\, n^n e^{-n}.$$