THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial for Week 7

MATH1903: Integral Calculus and Modelling (Advanced) — Semester 2, 2009

Lecturers: Holger Dullin and James Parkinson

Questions to attempt in class

1. (a) Compute the nth order Taylor polynomial of $f(x) = \ln(1+x)$ about $x = 0$.

(b) Use Taylor's Theorem to write down an expression for the remainder term.

(c) Deduce that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{for all } x \in [0,1].$$

2. (a) Compute the Taylor series for $\cos x$ about $x = 0$. Show that the Taylor series converges to $\cos x$ for all $x \in \mathbb{R}$.

(b) Write down the Taylor series for $\cos(x^3)$.
Note: It would be horrid to find the series for $\cos(x^3)$ *directly!*

3. (a) Use Taylor's Theorem to show that for all $x \geq 0$

$$1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 \leq \frac{1}{\sqrt{1+x}} \leq 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4.$$

(b) Hence give upper and lower bounds for the integral $\displaystyle\int_0^{1/2} \frac{1}{\sqrt{1+x^3}}\, dx$.

Questions for extra practice

4. The Taylor series for $\tan^{-1} x$ is not easy to find directly because the derivatives of $\tan^{-1} x$ get messy quickly. This question outlines an indirect method.

(a) Show that

$$\frac{1}{1+t^2} = \sum_{k=0}^{n-1} (-1)^k t^{2k} + \frac{(-1)^n t^{2n}}{1+t^2} \qquad \text{for all } t \in \mathbb{R}.$$

(b) Deduce that

$$\tan^{-1} x = \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k+1}}{2k+1} + E_n(x), \quad \text{where} \quad E_n(x) = (-1)^n \int_0^x \frac{t^{2n}}{1+t^2}\, dt.$$

(c) Show that $|E_n(x)| \leq \int_0^{|x|} t^{2n}\, dt = \frac{|x|^{2n+1}}{2n+1}$. Conclude that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \text{for all } -1 \leq x \leq 1.$$

5. (a) Compute the Taylor series of $f(x) = \sinh x$ about $x = 0$, and show that the series converges to $\sinh x$ for all $x \in \mathbb{R}$.

(b) Write down the Taylor series for $\sinh(x^2)$.

(c) Assuming that you can freely interchange the order of summation and integration, find series formulas for the integrals

$$\int_0^1 \sinh(x^2)\,dx \qquad \text{and} \qquad \int_0^1 \frac{\sinh x}{x}\,dx.$$

Questions for interest

6. From Question 1 we have the formula

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

Unfortunately this converges pathetically slowly - it turns out that you need 1565238 terms to get $\ln 2$ correct to 6 decimal places! We can do much better using the function

$$f(x) = \ln\left(\frac{1+x}{1-x}\right)$$

and noticing that $f(1/3) = \ln 2$.

(a) Calculate the Taylor series of $f(x)$ about $x = 0$.
Hint: Write $f(x) = \ln(1+x) - \ln(1-x)$.

(b) Use the Taylor polynomial $T_6(1/3)$ to approximate $\ln 2$. Estimate the size of the remainder term $R_6(1/3)$, and deduce that you have $\ln 2$ correct to 2 decimals.

7. From Question 4 we have *Leibnitz's Formula*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

This series is essentially useless for the purpose of approximating π (try it!). Luckily there is nice trick available. Recall the identity from the assignment:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), \qquad \text{valid for } xy < 1.$$

(a) Use the above identity to show that $4\tan^{-1}(1/5) = \tan^{-1}(120/19)$ and $\tan^{-1} 1 + \tan^{-1}(1/239) = \tan^{-1}(120/119)$.

(b) Hence prove *Machin's formula*:

$$\pi = 16\tan^{-1}\left(\frac{1}{5}\right) - 4\tan^{-1}\left(\frac{1}{239}\right).$$

Now use the first five terms from the $\tan^{-1}$ series from Question 4 to approximate π. Check your answer against your calculator.