The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial for Week 7

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2009

Lecturers: Holger Dullin and James Parkinson

Questions to attempt in class

1. (a) Compute the nth order Taylor polynomial of $f(x) = \ln(1+x)$ about $x = 0$.

Solution: The derivatives of $f(x)$ are:

$$\begin{aligned} f(x) &= \ln(1+x) & & f(0) = 0 \\ f^{(1)}(x) &= (1+x)^{-1} & & f^{(1)}(0) = 1 \\ f^{(2)}(x) &= -1!(1+x)^{-2} & & f^{(2)}(0) = -1! \\ f^{(3)}(x) &= 2!(1+x)^{-3} & \Longrightarrow & f^{(3)}(0) = 2! \\ f^{(4)}(x) &= -3!(1+x)^{-4} & & f^{(4)}(0) = -3! \\ &\vdots & & \vdots \\ f^{(n)}(x) &= (-1)^{n-1}(n-1)!(1+x)^{-n} & & f^{(n)}(0) = (-1)^{n-1}(n-1)! \end{aligned}$$

Therefore the nth order Taylor polynomial is

$$T_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}.$$

(b) Use Taylor's Theorem to write down an expression for the remainder term.

Solution: By Taylor's Theorem

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \qquad \text{for some } c \text{ between } 0 \text{ and } x.$$

Since $f^{(n+1)}(x) = (-1)^n n!(1+x)^{-n-1}$, we have

$$R_n(x) = (-1)^n \frac{x^{n+1}}{n+1}(1+c)^{-n-1} \quad \text{for some } c \text{ between } 0 \text{ and } x.$$

(c) Deduce that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{for all } x \in [0,1].$$

Solution: We need to show that $\lim_{n\to\infty} R_n(x) = 0$ for all $x \in [0,1]$. Since $0 \le c \le 1$ we have $(1+c)^{-n-1} \le 1$. Therefore for $x \in [0,1]$ we have

$$|R_n(x)| \le \frac{x^{n+1}}{n+1}.$$

This tends to zero for each fixed $x \in [0,1]$, and so the Taylor series converges to $f(x)$ for all $x \in [0,1]$.

Remark: The equality actually holds for all $-1 < x \le 1$.

2. (a) Compute the Taylor series for $\cos x$ about $x = 0$. Show that the Taylor series converges to $\cos x$ for all $x \in \mathbb{R}$.

Solution: We compute all the derivatives, and evaluate them at 0:

$$\begin{aligned} f(x) &= \cos x \\ f^{(1)}(x) &= -\sin x \\ f^{(2)}(x) &= -\cos x \\ f^{(3)}(x) &= \sin x \\ f^{(4)}(x) &= \cos x \\ &\vdots \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} f(0) &= 1 \\ f^{(1)}(0) &= 0 \\ f^{(2)}(0) &= -1 \\ f^{(3)}(0) &= 0 \\ f^{(4)}(0) &= 1 \\ &\vdots \end{aligned}$$

So we see that $f^{(k)}(0)$ repeats: $1, 0, -1, 0, 1, 0, -1, 0, \ldots$. Therefore the Taylor series is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .$$

At this stage we need to resist the temptation of writing "$\cos x =$ Taylor series". This is something we need to *prove*. To do this we need to show that

$$\lim_{n\to\infty} R_n(x) = 0 \qquad \text{for all } x \in \mathbb{R}.$$

By Taylor's Theorem we have

$$R_n(x) = \frac{x^{n+1}}{(n+1)!}\left(\frac{d^{n+1}}{dx^{n+1}}\cos x\right)\Bigg|_{x=c} \qquad \text{for some } c \text{ between } 0 \text{ and } x.$$

But the $(n+1)$-th derivative of $\cos x$ is one of $\cos x, -\cos x, \sin x, -\sin x$, and hence

$$\left|\frac{d^{n+1}}{dx^{n+1}}\cos x\right| \le 1 \qquad \text{for all } x \in \mathbb{R}.$$

Therefore

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \qquad \text{for all } x \in \mathbb{R},$$

and by the ratio test for sequences this converges to 0 for each fixed $x \in \mathbb{R}$. Hence $R_n(x) \to 0$ for each x, and so the Taylor series converges to $\cos x$ for all $x \in \mathbb{R}$. So now we can write the equality

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad \text{for all } x \in \mathbb{R}.$$

(b) Write down the Taylor series for $\cos(x^3)$.
Note: It would be horrid to find the series for $\cos(x^3)$ *directly!*

Solution: Just replace x by x^3 in the Taylor series for $\cos x$. The Taylor series is

$$1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \cdots ,$$

and this series converges to $\cos(x^3)$ for all $x \in \mathbb{R}$.

3. (a) Use Taylor's Theorem to show that for all $x \geq 0$

$$1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 \leq \frac{1}{\sqrt{1+x}} \leq 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4.$$

Solution: The strategy here is to write

$$f(x) = T_3(x) + R_3(x) \qquad \text{and} \qquad f(x) = T_4(x) + R_4(x)$$

and use Taylor's Theorem to show that $R_3(x) \geq 0$ and $R_4(x) \leq 0$.
First, let's compute the Taylor polynomials $T_3(x)$ and $T_4(x)$. You could use what you know about the binomial series from lectures, but let's just do it by hand in this question. We'll compute the first 5 derivatives (because we will need the 5th one for the remainder $R_4(x)$).

$$\begin{aligned}
f(x) &= (1+x)^{-1/2} & & & f(0) &= 1 \\
f^{(1)}(x) &= -\frac{1}{2}(1+x)^{-3/2} & & & f^{(1)}(0) &= -\frac{1}{2} \\
f^{(2)}(x) &= \frac{3}{4}(1+x)^{-5/2} & & & f^{(2)}(0) &= \frac{3}{4} \\
f^{(3)}(x) &= -\frac{15}{8}(1+x)^{-7/2} & &\Longrightarrow & f^{(3)}(0) &= -\frac{15}{8} \\
f^{(4)}(x) &= \frac{105}{16}(1+x)^{-9/2} & & & f^{(4)}(0) &= \frac{105}{16} \\
f^{(5)}(x) &= -\frac{945}{32}(1+x)^{-11/2} & & & &
\end{aligned}$$

Therefore

$$\begin{aligned}
T_3(x) &= 1 - \frac{1}{2}x + \frac{3}{4}\frac{x^2}{2!} - \frac{15}{8}\frac{x^3}{3!} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 \\
T_4(x) &= T_3(x) + \frac{105}{16}\frac{x^4}{4!} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4.
\end{aligned}$$

Taylor's Theorem gives

$$\begin{aligned}
R_3(x) &= \frac{105}{16}\frac{x^4}{4!}(1+c_1)^{-9/2} & &\text{for some } c_1 \text{ between 0 and } x \\
R_4(x) &= -\frac{945}{32}\frac{x^5}{5!}(1+c_2)^{-11/2} & &\text{for some } c_2 \text{ between 0 and } x.
\end{aligned}$$

Therefore if $x \geq 0$ we have $R_3(x) \geq 0$ and $R_4(x) \leq 0$. Hence

$$T_3(x) \leq T_3(x) + R_3(x) = f(x) = T_4(x) + R_4(x) \leq T_4(x),$$

which establishes the required inequalities.

(b) Hence give upper and lower bounds for the integral $\displaystyle\int_0^{1/2} \frac{1}{\sqrt{1+x^3}}\,dx$.

Solution: Replacing x by x^3 in the inequalities from (a) gives

$$\int_0^{1/2} T_3(x^3)\,dx \leq \int_0^{1/2} \frac{1}{\sqrt{1+x^3}}\,dx \leq \int_0^{1/2} T_4(x^3)\,dx.$$

We have

$$\int_0^{1/2} T_3(x^3)\,dx = \int_0^{1/2} \left(1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9\right)\,dx = 0.4925755\ldots$$

and

$$\int_0^{1/2} T_4(x^3)\,dx = \int_0^{1/2} \left(1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9 + \frac{35}{128}x^{12}\right)\,dx = 0.4925780\ldots$$

Therefore the integral equals $0.49257\ldots$ with the first 5 places correct.

Questions for extra practice

4. The Taylor series for $\tan^{-1} x$ is not easy to find directly because the derivatives of $\tan^{-1} x$ get messy quickly. This question outlines an indirect method.

(a) Show that

$$\frac{1}{1+t^2} = \sum_{k=0}^{n-1} (-1)^k t^{2k} + \frac{(-1)^n t^{2n}}{1+t^2} \qquad \text{for all } t \in \mathbb{R}.$$

Solution: This is just a rearrangement of the geometric sum formula

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r} = \frac{1}{1-r} - \frac{r^n}{1-r}$$

with $r = -t^2$. The formula is valid whenever $r \neq 1$. That is, $t^2 \neq -1$, and so the formula is valid for all $t \in \mathbb{R}$.

(b) Deduce that

$$\tan^{-1} x = \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k+1}}{2k+1} + E_n(x), \quad \text{where} \quad E_n(x) = (-1)^n \int_0^x \frac{t^{2n}}{1+t^2}\,dt.$$

Solution: Integrate the formula from (a) between 0 and x.

(c) Show that $|E_n(x)| \leq \int_0^{|x|} t^{2n}\,dt = \frac{|x|^{2n+1}}{2n+1}$. Conclude that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \text{for all } -1 \leq x \leq 1.$$

Solution: Notice that $\frac{1}{1+t^2} \leq 1$ for all $t \in \mathbb{R}$. If $x \geq 0$ then we get

$$|E_n(x)| = \int_0^x \frac{t^{2n}}{1+t^2}\,dt \leq \int_0^x t^{2n}\,dt = \frac{x^{2n+1}}{2n+1}.$$

If $x < 0$

$$E_n(x) = (-1)^n \int_0^x \frac{t^{2n}}{1+t^2}\,dt = -(-1)^n \int_x^0 \frac{t^{2n}}{1+t^2}\,dt = -(-1)^n \int_0^{-x} \frac{t^{2n}}{1+t^2}\,dt,$$

where we have used the fact that the integrand is even. Hence if $x < 0$ we have

$$|E_n(x)| = \int_0^{|x|} \frac{t^{2n}}{1+t^2}\,dt \leq \int_0^{|x|} t^{2n}\,dt = \frac{|x|^{2n+1}}{2n+1}.$$

Therefore if $|x| \leq 1$ we have $E_n(x) \to 0$ as $n \to \infty$. This establishes the formula

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \text{for all } -1 \leq x \leq 1.$$

5. (a) Compute the Taylor series of $f(x) = \sinh x$ about $x = 0$, and show that the series converges to $\sinh x$ for all $x \in \mathbb{R}$.

Solution: Calculating the derivatives gives

$$\begin{aligned} f(x) &= \sinh x & & & f(0) &= 0 \\ f^{(1)}(x) &= \cosh x & & & f^{(1)}(0) &= 1 \\ f^{(2)}(x) &= \sinh x & &\Longrightarrow & f^{(2)}(0) &= 0 \\ f^{(3)}(x) &= \cosh x & & & f^{(3)}(0) &= 1 \\ &\vdots & & & &\vdots \end{aligned}$$

and so $f^{(n)}(0)$ has pattern $0, 1, 0, 1, 0, 1, \ldots$ Therefore the Taylor series is (as you no doubt knew):

$$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

Let's show that this series converges to $\sinh x$ for all $x \in \mathbb{R}$. To do this we show that the remainder term $R_n(x)$ tends to 0 for all $x \in \mathbb{R}$, and to do this we use Taylor's Theorem to write down a formula for the remainder term:

$$R_n(x) = \frac{x^{n+1}}{(n+1)!}\left(\frac{d^{n+1}}{dx^{n+1}}\sinh x\right)\Bigg|_{x=c} \qquad \text{for some } c \text{ between 0 and } x.$$

But the $(n+1)$-th derivative of $\sinh x$ is either $\sinh x$ or $\cosh x$. Using the definitions of sinh and cosh gives the inequalities

$$|\sinh x| \leq e^{|x|} \qquad \text{and} \qquad |\cosh x| \leq e^{|x|} \qquad \text{for all } x \in \mathbb{R},$$

and therefore

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}e^{|c|} \quad \text{for some } c \text{ between 0 and } x.$$

Since c is between 0 and x we have $0 \leq |c| \leq |x|$, and hence $e^{|c|} \leq e^{|x|}$. Therefore

$$|R_n(x)| \leq e^{|x|}\frac{|x|^{n+1}}{(n+1)!}.$$

Therefore the remainder tends to 0 as $n \to \infty$ (for each fixed x) by the ratio test for sequences. This shows that the Taylor series converges to $\sinh x$ for all $x \in \mathbb{R}$.

(b) Write down the Taylor series for $\sinh(x^2)$.

Solution: Just replace x by x^2 in the Taylor series for $\sinh x$. So the Taylor series is

$$x^2 + \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{4n+2}}{(2n+1)!}.$$

By the previous part, this series converges to $\sinh(x^2)$ for all $x \in \mathbb{R}$.

(c) Assuming that you can freely interchange the order of summation and integration, find series formulas for the integrals

$$\int_0^1 \sinh(x^2)\,dx \qquad \text{and} \qquad \int_0^1 \frac{\sinh x}{x}\,dx.$$

Solution: We have

$$\begin{aligned}\int_0^1 \sinh(x^2)\,dx &= \int_0^1 \left(\sum_{n=0}^{\infty} \frac{x^{4n+2}}{(2n+1)!}\right) dx \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \int_0^1 x^{4n+2}\,dx \\ &= \sum_{n=0}^{\infty} \frac{1}{(4n+3)(2n+1)!}.\end{aligned}$$

Similarly, using $\frac{\sinh x}{x} = 1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots$ gives

$$\int_0^1 \frac{\sinh x}{x}\,dx = \int_0^1 \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!}\right) dx = \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+1)!}.$$

Both of these series converge very quickly, and so can be used to give accurate approximations to the integrals.

Remark: You'll learn more about interchanging the order of integration and summation in later mathematics courses.

Questions for interest

6. From Question 1 we have the formula

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

Unfortunately this converges pathetically slowly - it turns out that you need 1565238 terms to get $\ln 2$ correct to 6 decimal places! We can do much better using the function

$$f(x) = \ln\left(\frac{1+x}{1-x}\right)$$

and noticing that $f(1/3) = \ln 2$.

(a) Calculate the Taylor series of $f(x)$ about $x = 0$.
Hint: Write $f(x) = \ln(1+x) - \ln(1-x)$.

Solution: Writing $f(x) = \ln(1+x) - \ln(1-x)$ makes it clear that

$$f^{(n)}(x) = \frac{d^n}{dx^n}\ln(1+x) - \frac{d^n}{dx^n}\ln(1-x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n} + \frac{(n-1)!}{(1-x)^n}.$$

Evaluating at $x = 0$, we get

$$f^{(n)}(0) = (n-1)!\big((-1)^{n-1} + 1\big),$$

and so

$$\frac{f^{(n)}(0)}{n!} = \frac{1 + (-1)^{n-1}}{n} = \begin{cases} 2/n & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Hence the Taylor series for $f(x)$ is

$$2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n-1}}{2n-1} + \cdots\right).$$

(b) Use the Taylor polynomial $T_6(1/3)$ to approximate $\ln 2$. Estimate the size of the remainder term $R_6(1/3)$, and deduce that you have $\ln 2$ correct to 2 decimals.

Solution: The Taylor polynomial $T_6(x)$ is

$$T_6(x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5}\right)$$

(the coefficient of x^6 is zero). Therefore $T_6(1/3) = 0.693004115\ldots$
We will now approximate the error. By Taylor's Theorem we have

$$R_6(1/3) = \frac{f^{(7)}(c)}{7!}(1/3)^7 \qquad \text{for some } c \text{ between } 0 \text{ and } 1/3,$$

and so by the formula for $f^{(n)}(x)$ given above we have

$$R_6(1/3) = \frac{1}{3^7 \cdot 7}\left(\frac{1}{(1+c)^7} + \frac{1}{(1-c)^7}\right) \quad \text{for some } 0 \le c \le 1/3.$$

Therefore $R_6(1/3) \ge 0$, and since $0 \le c \le 1/3$ we have

$$\frac{1}{(1+c)^7} \le 1 \qquad \text{and} \qquad \frac{1}{(1-c)^7} \le \left(\frac{3}{2}\right)^7.$$

Putting this together gives

$$0 \le R_6(1/3) \le \frac{1}{7}\left(\frac{1}{3^7} + \frac{1}{2^7}\right) = 0.00118139\ldots$$

Therefore we know that $\ln 2 = 0.69\ldots$ with the first 2 places correct. To get 6 decimals correct you only need to use 6 terms in the series. This is a lot better than 1565238 terms!

7. From Question 4 we have *Leibnitz's Formula*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

This series is essentially useless for the purpose of approximating π (try it!). Luckily there is nice trick available. Recall the identity from the assignment:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), \qquad \text{valid for } xy < 1.$$

(a) Use the above identity to show that $4\tan^{-1}(1/5) = \tan^{-1}(120/19)$ and $\tan^{-1} 1 + \tan^{-1}(1/239) = \tan^{-1}(120/119)$.

Solution: We have

$$2\tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{5}}{1-\frac{1}{25}}\right) = \tan^{-1}\left(\frac{5}{12}\right),$$

and therefore

$$4\tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{5}{12}\right) + \tan^{-1}\left(\frac{5}{12}\right) = \tan^{-1}\left(\frac{\frac{5}{12}+\frac{5}{12}}{1-\frac{25}{144}}\right) = \tan^{-1}\left(\frac{120}{119}\right).$$

We also have

$$\tan^{-1}(1) + \tan^{-1}\left(\frac{1}{239}\right) = \tan^{-1}\left(\frac{1+\frac{1}{239}}{1-\frac{1}{239}}\right) = \tan^{-1}\left(\frac{120}{119}\right).$$

(b) Hence prove *Machin's formula*:

$$\pi = 16\tan^{-1}\left(\frac{1}{5}\right) - 4\tan^{-1}\left(\frac{1}{239}\right).$$

Now use the first five terms from the $\tan^{-1}$ series from Question 4 to approximate π. Check your answer against your calculator.

Solution: From the previous part we have

$$\frac{\pi}{4} = \tan^{-1}(1) = \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{1}{239}\right) = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right).$$

Now multiply by 4.
Using the Taylor series for $\tan^{-1} x$ (with $x = \frac{1}{5}$ and $x = \frac{1}{239}$) we approximate

$$\begin{aligned}\pi &\approx 16\left(\frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} - \frac{1}{7\cdot 5^7} + \frac{1}{9\cdot 5^9}\right)\\ &\quad - 4\left(\frac{1}{239} - \frac{1}{3\cdot 239^3} + \frac{1}{5\cdot 239^5} - \frac{1}{7\cdot 239^7} + \frac{1}{9\cdot 239^9}\right)\\ &= 3.141592652615\ldots\end{aligned}$$

The first 8 decimals are correct. You would need millions of terms in Leibnitz's formula to obtain this accuracy.

Remark: There actually was a time before pocket calculators. A time when only a few decimal places of π were known. Now-days some of this "computing stuff by hand" seems a tad outdated, but think of how happy people (mathematicians, engineers, physicists - I guess they were all the same back then) must have been when John Machin discovered his quickly converging series for π in 1706. He used this series to compute π to 100 decimal places by hand (fun guy to have at a dinner party). At the time this was the 'world record'. In 1945 only 527 decimals were known - so Machin really did a pretty good job. Now more than 2,000,000,000,000 decimals are known. This kind of accuracy is a bit overwhelming: Knowing π to 50 decimal places is sufficient to compute the circumference (given the diameter) of the known universe to within the thickness of an electron.