Notes on Integral Calculus and Modelling
2nd Instalment

Contents

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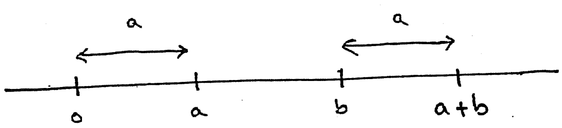
2.1

Logs and exponentials

Let $a, b > 0$.

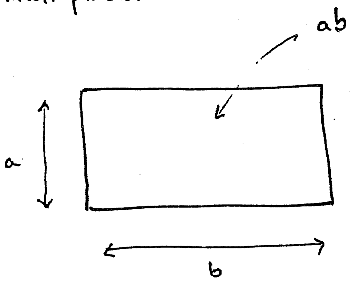
Easy to understand:

- addition



translation along the real line

- multiplication



area of a rectangle

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But how does one define or conceptualize exponentiation ?!

$\boxed{\text{Example : What is } a^{\pi} \text{ ?}}$

$$a^n = \underbrace{a \times a \times \dots \times a}_{n \text{ times}} \quad \text{if } n \in \mathbb{Z}^+$$

$$a^{1/m} = c \quad \text{where}$$

$$\underbrace{c \times c \times \dots \times c}_{m \text{ times}} = a \quad \text{if } m \in \mathbb{Z}^+$$

$$a^{n/m} = (a^{1/m})^n = c^n$$

so we have $\boxed{a^q \text{ for any } q \in \mathbb{Q}^+}$

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Easy fact : For all $q_1, q_2 \in \mathbb{Q}^+$,

$$q_1 < q_2 \implies a^{q_1} < a^{q_2}.$$

assuming $a > 1$

Let

$$q_1 < q_2 < \dots < q_n < \dots < \pi < 4$$

where each $q_i \in \mathbb{Q}^+$ and

$$\pi = \lim_{n \to \infty} q_n$$

(e.g. use the decimal expansion of π).

Then

$$\underbrace{a^{q_1} < a^{q_2} < \dots < a^{q_n} < \dots}_{\text{monotonic sequence}} < a^4$$

upper bound

so $\boxed{\lim_{n \to \infty} a^{q_n} \text{ exists !!}}$

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Reason :

Monotone Convergence Theorem :

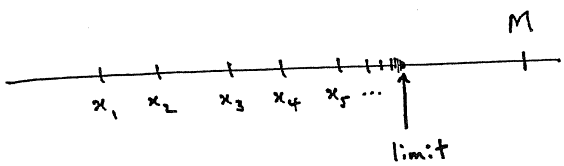
Let

$$x_1 \le x_2 \le x_3 \le \dots \le x_n \le \dots \le M$$

be an infinite nondecreasing sequence of real numbers bounded above by M.

Then $\lim_{n \to \infty} x_n$ exists.

idea : eventually the numbers "bunch up"



Proof: Put $X = \{x_n \mid n \in \mathbb{Z}^+\}$.

Then X is bounded above by M, so by completeness of $\mathbb{R}$

X has a least upper bound L

Completeness of $\mathbb{R}$ says: any nonempty set of reals which is bounded above has a least upper bound.

Claim: $\lim\limits_{n \to \infty} x_n = L$

We have to prove

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{Z}^+)(\forall n \geq N) \quad |x_n - L| < \varepsilon .$$

Let $\varepsilon > 0$.

If $x_n \leq L - \varepsilon$ for all n then $L - \varepsilon$ is an upper bound for X smaller than L, contradicting that L is the least upper bound.

Hence $L - \varepsilon < x_N \leq L$ for some N.

Then

$$L - \varepsilon < x_N \leq x_{N+1} \leq x_{N+2} \leq \dots \leq L$$

so $|x_n - L| < \varepsilon \quad \forall n \geq N$.

This proves

$$\lim_{n \to \infty} x_n = L . \quad \square$$

Returning to our quest for a^{π}: we have

$$q_1 < q_2 < \dots < q_n < \dots < \pi < 4$$

where $\lim\limits_{n \to \infty} q_n = \pi$

and $a^{q_1} < a^{q_2} < \dots < a^{q_n} < \dots < a^4$

so $\lim\limits_{n \to \infty} a^{q_n}$ exists.

Define

$$a^{\pi} = \lim_{n \to \infty} a^{q_n}$$

This method allows us to define a^b for any $b \in \mathbb{R}$ by using limits of sequences using rational exponents !!!

Completely different approach !!!

We will define

$$\boxed{a^b = e^{b \ln a}}$$

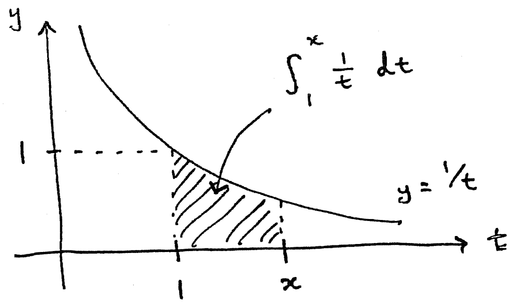
provided we can make sense of

- $\ln a$ "the natural logarithm of a"
- the real number e
- ~~arbitrary powers of e~~

Advantage of this method: "constructive" rather than "existential"

Gap in following table ?!

powers of x	an antiderivative
x^3	$x^4/4$
x^2	$x^3/3$
x	$x^2/2$
$1 = x^0$	$x = x^1/1$
$1/x = x^{-1}$	? ?
$1/x^2 = x^{-2}$	$-\frac{1}{x} = \frac{x^{-1}}{-1}$
$1/x^3 = x^{-3}$	$-\frac{1}{2x^2} = \frac{x^{-2}}{-2}$
x^n	$\frac{x^{n+1}}{n+1}$ if $n \neq -1$



By the Fundamental Theorem of Calculus (part 1)

$$\boxed{\frac{d}{dx} \int_1^x \frac{1}{t}\, dt = \frac{1}{x}}$$

Define the <u>natural logarithmic function</u> ln by, for $x > 0$,

$$\boxed{\ln x = \int_1^x \frac{1}{t}\, dt}\ .$$

<u>Properties</u> :

$$\boxed{\ln 1 = \int_1^1 \frac{1}{t}\, dt = 0}$$

If $x > 1$ then

$$\ln x = \int_1^x \frac{1}{t}\, dt > 0$$

(positive area)

If $0 < x < 1$ then

$$\ln x = \int_1^x \frac{1}{t}\, dt = -\int_x^1 \frac{1}{t}\, dt < 0 .$$

<u>The derivative</u> :

$$\boxed{\text{If } x > 0 \text{ then } \frac{d}{dx} \ln x = \frac{1}{x}.} \quad (*)$$

If $x < 0$ then $-x > 0$

and
$$\frac{d}{dx} \ln(-x) = \frac{d}{du} \ln(u) \frac{du}{dx}$$

where $u = -x$

$$= \frac{1}{u}(-1)$$

$$= -\frac{1}{-x} = \frac{1}{x} .$$

$$\boxed{\text{If } x < 0 \text{ then } \frac{d}{dx} \ln(-x) = \frac{1}{x}.} \quad (**)$$

Combining (*) and (**) :

$$\boxed{\text{If } x \neq 0 \text{ then } \frac{d}{dx} \ln|x| = \frac{1}{x}.}$$

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Logarithms "turn products into sums":

$$\boxed{\text{For } a, b > 0 \qquad \ln(ab) = \ln a + \ln b.}$$

Proof: Fix $a > 0$ and define

$$g(x) = \ln(ax) \quad \text{for } x > 0.$$

Then $g'(x) = \dfrac{d}{dx}\ln(ax)$

$$= \left(\frac{d}{du}\ln u\right)\left(\frac{du}{dx}\right)$$

where $u = ax$

$$= \frac{1}{u}\cdot a$$

$$= \frac{a}{ax} = \frac{1}{x}.$$

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Hence

$$g(x) = \ln x + C$$

for some constant C.

But $g(1) = \ln(a\cdot 1) = \ln a$

$$= \ln 1 + C$$

$$= 0 + C = C,$$

so $C = \ln a$.

Hence $g(x) = \ln x + \ln a$.

In particular

$$\ln(ab) = g(b) = \ln b + \ln a$$

$$= \ln a + \ln b,$$

as required.

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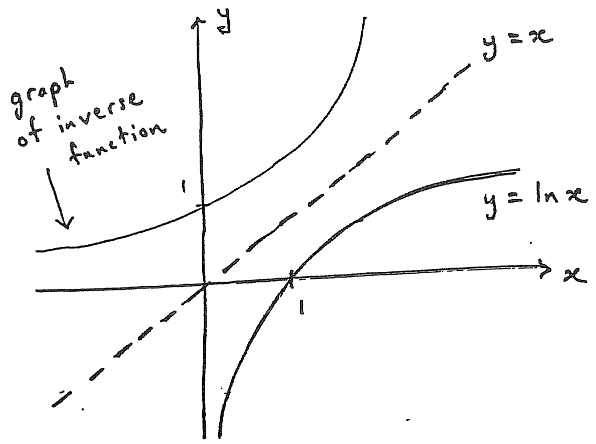
The exponential function

$$\frac{d}{dx}\ln x = \frac{1}{x} > 0 \quad \text{for } x > 0$$

so $\ln$ is an <u>increasing</u> function.



Facts (tricky, proofs below)

$$\boxed{\lim_{x\to\infty}\ln x = \infty}, \quad \boxed{\lim_{x\to 0^+}\ln x = -\infty}.$$

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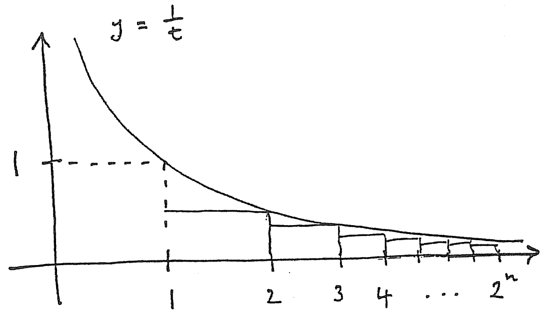

(not to scale!)

Using a lower Riemann sum we get

$$\int_1^{2^n}\frac{dt}{t} \geq \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^n}$$

$$= \underbrace{\frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}$$

$$+ \cdots + \underbrace{\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}}$$

$$\geq \underbrace{\frac{1}{2}} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}$$

$$+ \cdots + \underbrace{\frac{1}{2^n} + \cdots + \frac{1}{2^n}}$$

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So

$$\int_1^{2^n} \frac{dt}{t} \geqslant \underbrace{\frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{n \text{ copies}}$$

$$= \frac{n}{2}$$

Thus

$$\boxed{\ln(2^n) \geqslant \frac{n}{2}}$$

But

$$\lim_{n \to \infty} \frac{n}{2} = \infty$$

so

$$\lim_{n \to \infty} \ln(2^n) = \infty$$

(Squeeze Law)

so

$$\lim_{m \to \infty} \ln(m) = \infty \qquad (m \in \mathbb{Z}^+)$$

so

$$\boxed{\lim_{x \to \infty} \ln x = \infty.} \qquad (x \in \mathbb{R}^+)$$

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$$\boxed{\text{Corollary: } \lim_{x \to 0^+} \ln x = -\infty.}$$

Proof:

$$\lim_{x \to 0^+} \ln x = \lim_{y \to \infty} \ln\left(\frac{1}{y}\right)$$

$$= \lim_{x \to \infty} \ln\left(\frac{1}{x}\right)$$

$$= \lim_{x \to \infty} (-\ln x)$$

(Why?)

$$= -\lim_{x \to \infty} \ln x$$

$$= -\infty.$$

□

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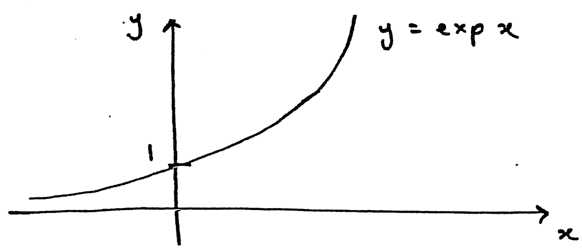
$$\boxed{\begin{array}{l} \text{range of } \ln = \mathbb{R} \\ \text{domain of } \ln = (0, \infty) \end{array}}$$

The inverse function of ln is called the <u>exponential function</u>, denoted by exp.

Thus

$$\boxed{\begin{array}{l} \text{domain of } \exp = \mathbb{R} \\ \text{range of } \exp = (0, \infty) \end{array}}$$



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Thus

$$\boxed{\begin{array}{ll} \exp(\ln x) = x & \text{for } x > 0 \\ \ln(\exp x) = x & \text{for } x \in \mathbb{R} \end{array}}$$

$$\boxed{\text{exp and ln "undo each other".}}$$

<u>Properties of exp</u>:

$$\boxed{\exp(0) = 1} \qquad (\text{since } \ln 1 = 0)$$

$$\boxed{\exp(x) > 0 \quad \text{for all } x \in \mathbb{R}}$$

$$\boxed{\exp(a+b) = [\exp(a)][\exp(b)]}$$

– called the <u>exponential law</u>.

Proof of the exponential law:

Let $a, b \in \mathbb{R}$ and

put $x = \exp(a)$, $y = \exp(b)$.

Then $a = \ln(x)$, $b = \ln(y)$

and

$$\exp(a+b) = \exp(\ln(x) + \ln(y))$$

$$= \exp(\ln(xy))$$

by a property of ln

$$= xy$$

since exp "undoes" ln

$$= \exp(a)\exp(b),$$

as required.

Most important property:

$$\boxed{\frac{d}{dx}\exp(x) = \exp(x)}$$

Proof: $x = \ln(\exp(x))$

so

$$1 = \frac{dx}{dx} = \frac{d}{dx}\left(\ln(\exp(x))\right)$$

$$= \frac{d}{du}(\ln(u))\frac{du}{dx}$$

where $u = \exp(x)$

$$= \frac{1}{u}\frac{du}{dx},$$

whence

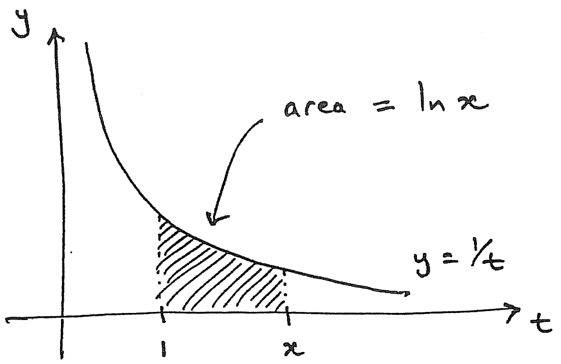
$$\frac{d}{dx}\exp(x) = \frac{du}{dx} = u = \exp(x),$$

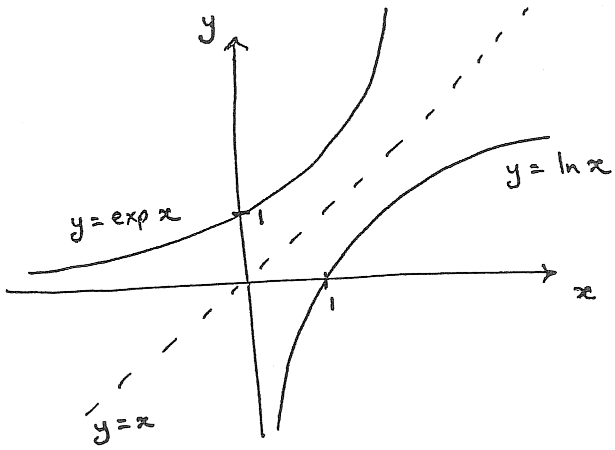
as required.

Recapping, we have the

natural logarithm

$$\boxed{\ln x = \int_1^x \frac{1}{t}\,dt \quad \text{for } x > 0}$$





Recall the exponential function

exp is the inverse of ln,

so, for $a > 0$, $b \in \mathbb{R}$

$$\boxed{b = \ln(a) \iff \exp(b) = a}$$

Important properties :

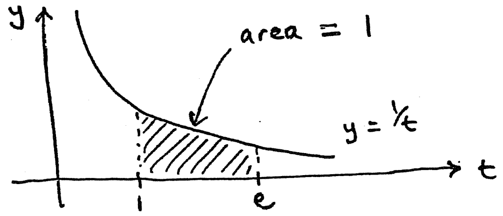
(1) $\ln(a \cdot b) = \ln a + \ln b$

(2) $\exp(a+b) = \exp(a) \cdot \exp(b)$

(3) $\dfrac{d}{dx} \ln|x| = \dfrac{1}{x}$

(4) $\dfrac{d}{dx} \exp(x) = \exp(x)$.

Put $e = \exp(1)$, so $\ln e = 1$



We want

$$\boxed{\exp(x) = e^{x}}$$

yet to be defined

so that the variable x is an exponent.

Observe, for n positive integer,

$$\ln(a^n) = \ln(\underbrace{a \cdot a \ldots a}_{n \text{ times}})$$

$$= \underbrace{\ln a + \ln a + \cdots + \ln a}_{n \text{ times}}$$

$$= n \ln(a) .$$

For n positive integer

$$\boxed{\ln(a^n) = n \ln(a)}$$

so

$$\boxed{a^n = \exp(n \ln(a))} .$$

This suggests the following definition :

For $x \in \mathbb{R}$ and $a > 0$ define

$$\boxed{a^x = \exp(x \ln(a))} .$$

In particular

$$\boxed{e^x = \exp(x \ln e) = \exp(x)} .$$

Properties :

(1) $(ab)^c = a^c b^c$

(2) $a^c a^d = a^{c+d}$

(3) $(a^c)^d = a^{cd}$

(4) $a^0 = 1$

(5) $a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$ if $n \in \mathbb{Z}^+$

(6) $\ln(a^x) = x \ln(a)$

(7) $\dfrac{d}{dx} x^a = a x^{a-1}$

Proofs : left as exercises. $\square$

Further techniques of integration

Integration by parts :

Recall the product rule

$$\boxed{\frac{d(uv)}{dx} = u\frac{dv}{dx} + \frac{du}{dx}v}$$

where u, v functions of x.

Antidifferentiate both sides with respect to x :

$$uv = \int\left[\frac{d(uv)}{dx}\right]dx$$

$$= \int\left[u\frac{dv}{dx} + \frac{du}{dx}v\right]dx$$

$$= \int\left[u\frac{dv}{dx}\right]dx + \int\left[\frac{du}{dx}v\right]dx$$

$$= \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

$$= \int u\,dv + \int v\,du$$

Rearranging yields the integration by parts formula :

$$\boxed{\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx}$$

or, more simply,

$$\boxed{\int u\,dv = uv - \int v\,du}.$$

Example :

$$\int \underset{u}{x}\,\underset{\frac{dv}{dx}}{e^x}\,dx = \underset{u}{x}\,\underset{v}{e^x} - \int \underset{v}{e^x}\,\underset{\frac{du}{dx}}{1}\,dx$$

$$= xe^x - \int e^x\,dx$$

$$= xe^x - e^x + C$$

Trying $u = e^x$, $\frac{dv}{dx} = x$ would make things more complicated !

Example:

$$\int \underset{u}{x^2}\, \underset{\frac{dv}{dx}}{e^x}\, dx = \underset{u}{x^2}\, \underset{v}{e^x} - \int \underset{v}{e^x}\, \underset{\frac{du}{dx}}{(2x)}\, dx$$

$$= x^2 e^x - 2\int x e^x\, dx$$

$$= x^2 e^x - 2(xe^x - e^x) + C$$

$$= e^x(x^2 - 2x + 2) + C.$$

Handy trick: Sometimes $\frac{dv}{dx} = 1$ helps.

Example:

$$\int \ln x\, dx = \int \underset{\frac{dv}{dx}}{1}.\underset{u}{\ln x}\, dx$$

$$= \underset{v}{x}\, \underset{u}{\ln x} - \int \underset{v}{x}\, \underset{\frac{du}{dx}}{\left(\tfrac{1}{x}\right)}\, dx$$

$$= x\ln x - \int 1\, dx$$

$$= x\ln x - x + C.$$

Another trick: use parts to express an integral in terms of itself and rearrange.

Example:

$$\int \underset{\frac{dv}{dx}}{e^x}\, \underset{u}{\sin x}\, dx = \underset{v}{e^x}\, \underset{u}{\sin x} - \int \underset{v}{e^x}\, \underset{\frac{du}{dx}}{\cos x}\, dx$$

$$= e^x \sin x - \int \underset{\frac{dv}{dx}}{e^x}\, \underset{u}{\cos x}\, dx$$

$$= e^x \sin x - \left(\underset{v}{e^x}\, \underset{u}{\cos x} - \int \underset{v}{e^x}\, \underset{\frac{du}{dx}}{(-\sin x)}\, dx \right)$$

$$= e^x(\sin x - \cos x) - \int e^x \sin x\, dx.$$

$$\boxed{\text{Put } I = \int e^x \sin x\, dx}$$

Then

$$I = e^x(\sin x - \cos x) - I$$

so $$2I = e^x(\sin x - \cos x) + C'$$

so $$\boxed{I = \frac{e^x}{2}(\sin x - \cos x) + C}.$$

For definite integrals, use

$$\boxed{\int_a^b u \frac{dv}{dx}\, dx = \Big[uv\Big]_a^b - \int_a^b v \frac{du}{dx}\, dx}$$

<u>Example</u>:

$$\int_1^e \underset{\frac{dv}{dx}}{\underset{\uparrow}{x}}\, \underset{u}{\underset{\uparrow}{\ln x}}\, dx = \left[\underset{v}{\underset{\uparrow}{\frac{x^2}{2}}}\, \underset{u}{\underset{\uparrow}{\ln x}}\right]_1^e - \int_1^e \underset{v}{\underset{\uparrow}{\frac{x^2}{2}}}\, \underset{\frac{du}{dx}}{\underset{\uparrow}{\frac{1}{x}}}\, dx$$

$$= \frac{e^2}{2} - 0 - \frac{1}{2}\int_1^e x\, dx$$

$$= \frac{e^2}{2} - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^e$$

$$= \frac{e^2}{2} - \frac{1}{2}\left(\frac{e^2}{2} - \frac{1}{2}\right)$$

$$= \frac{e^2}{4} + \frac{1}{4}.$$

<u>Reduction formulae</u>:

These are <u>recursive</u> formulae, allowing calculation in several steps
- typically by reducing powers in an integrand

Commonly, reduction formulae are derived using integration by parts.

<u>Example</u>: Develop a formula for $\int \sin^n x\, dx$

<u>Solution</u>:

$$\int \sin^n x\, dx = \int \underset{u}{\underset{\uparrow}{\sin^{n-1} x}}\, \underset{\frac{dv}{dx}}{\underset{\uparrow}{\sin x}}\, dx$$

$$= \underset{u}{\underset{\uparrow}{\sin^{n-1} x}}\, \underset{v}{\underset{\uparrow}{(-\cos x)}} - \int \underset{v}{\underset{\uparrow}{(-\cos x)}}\, \underset{\frac{du}{dx}}{\underbrace{(n-1)\sin^{n-2} x\, (\cos x)}}\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \cos^2 x \sin^{n-2} x\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int (1-\sin^2 x)\sin^{n-2} x\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x - \sin^n x\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\left[\int \sin^{n-2} x\, dx - \int \sin^n x\, dx\right]$$

$$\boxed{\text{Put } I_n = \int \sin^n x\, dx}$$

Then

$$I_n = -\sin^{n-1} x \cos x + (n-1)\left[I_{n-2} - I_n \right]$$

$$= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n .$$

Hence

$$n I_n = I_n + (n-1) I_n$$

$$= -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

whence

$$\boxed{I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}}$$

Partial fractions and rational functions

A rational function is a quotient (ratio) of polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$, $Q(x)$ are polynomials.

If $\begin{cases} P(x) \text{ is constant} \\ Q(x) = (ax+b)^n \end{cases}$

or $\begin{cases} P(x) \text{ is linear} \\ Q(x) = (ax^2+bx+c)^n \end{cases}$

then it is possible to antidifferentiate using techniques so far discussed.

Otherwise, we use the method of partial fractions to "decompose" the rational function into pieces which are of this form.

Fundamental Theorem of Algebra:

Every polynomial $p(x)$ with coefficients from

$$\mathbb{C} = \{\text{complex numbers}\}$$

can be factorized into linear factors

$$p(x) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$$

for some $\underbrace{\lambda_1, \lambda_2, \ldots, \lambda_n}_{\text{called roots}} \in \mathbb{C}$.

If the coefficients of $p(x)$ come from $\mathbb{R}$ then the roots come in complex conjugate pairs

$$\lambda_1 = a+ib, \quad \lambda_2 = a-ib, \quad \ldots$$

If $b \neq 0$ then we get an irreducible quadratic factor:

$$(x-\lambda_1)(x-\lambda_2) = x^2 - 2ax + a^2+b^2$$

Consequence: all real polynomials factorize into linear and irreducible quadratic factors.

This leads to the following method:

Method for decomposing $\dfrac{P(x)}{Q(x)}$:

(1) Divide through by $Q(x)$

if

degree of $P(x)$

$\geqslant$ degree of $Q(x)$.

(2) Factorize $Q(x)$ into linear and irreducible quadratic factors.

(3) If $(x-a)$ is a factor, include a term $\dfrac{A}{x-a}$.

(4) If $(x-a)^n$ is a repeated factor, include terms

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}$$

(5) If x^2+bx+c is an irreducible quadratic factor, include a term

$$\frac{Ax+B}{x^2+bx+c}$$

(6) analogous to (4) if $(x^2+bx+c)^n$ is a repeated factor.

To find all constants that arise in (3), (4), (5), (6),

put everything over a common denominator and equate numerators.

Either

(i) comparing coefficients of powers of x

or

(ii) using convenient substitutions for x

enables constants to be found.

Example: Find

$$\int \frac{dx}{(x-1)(x-2)(x-3)}$$

Solution: We find A, B, C such that

$$\frac{1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

giving

$$1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$$

This must hold for all x, by continuity of polynomials !!!

Judicious choices of x yield A, B, C quickly.

Put $x=1$: $1 = A(-1)(-2)$, so $A = \frac{1}{2}$

$x=2$: $1 = B(1)(-1)$, so $B = -1$

$x=3$: $1 = C(2)(1)$, so $C = \frac{1}{2}$.

Thus

$$\frac{1}{(x-1)(x-2)(x-3)} = \frac{\frac{1}{2}}{x-1} + \frac{-1}{x-2} + \frac{\frac{1}{2}}{x-3}$$

so

$$\int \frac{dx}{(x-1)(x-2)(x-3)}$$

$$= \frac{1}{2}\int \frac{dx}{x-1} - \int \frac{dx}{x-2} + \frac{1}{2}\int \frac{dx}{x-3}$$

$$= \frac{1}{2}\ln|x-1| - \ln|x-2| + \frac{1}{2}\ln|x-3| + C$$

$$= \ln \frac{\sqrt{(x-1)(x-3)}}{|x-2|} + C .$$

Example : Find $\int \frac{dx}{x(x-1)^2}$.

Solution : Put

$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

so $\quad 1 = A(x-1)^2 + Bx(x-1) + Cx$.

Put $x=0$: $1 = A(-1)^2$, so $A = 1$.

$x=1$: $1 = C(1)$, so $C = 1$.

$x=2$: $1 = A + 2B + 2C$, so $B = -1$.

Hence $\int \frac{dx}{x(x-1)^2}$

$$= \int \frac{dx}{x} - \int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2}$$

$$= \ln|x| - \ln|x-1| - \frac{1}{x-1} + C$$

$$= \ln\left|\frac{x}{x-1}\right| - \frac{1}{x-1} + C .$$

Example : Find $\int \frac{x^4+x-1}{x^3+x}\,dx$.

Solution :

$$\begin{array}{r|l} & x \\ \hline x^3+x & x^4+x-1 \\ & x^4+x^2 \\ \hline & -x^2+x-1 \end{array}$$

so

$$\frac{x^4+x-1}{x^3+x} = x + \frac{-x^2+x-1}{x^3+x} .$$

Put $\frac{-x^2+x-1}{x^3+x} = \frac{-x^2+x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

so $\quad -x^2+x-1 = A(x^2+1) + (Bx+C)x$.

Put $x=0$: $-1 = A(1)$, so $A = -1$

giving $\quad -x^2+x-1 = -x^2-1 + Bx^2 + Cx$

so $\quad x = Bx^2 + Cx$.

Equating coefficients gives

$B = 0$, $C = 1$.

Hence

$$\int \frac{x^4+x-1}{x^3+x}\,dx$$

$$= \int x\,dx - \int \frac{dx}{x} + \int \frac{dx}{x^2+1}$$

$$= \frac{x^2}{2} - \ln|x| + \tan^{-1}x + C .$$

Improper integrals

The definite integral

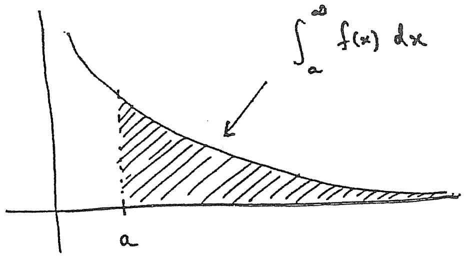
$$\int_a^b f(x)\,dx$$

was developed assuming

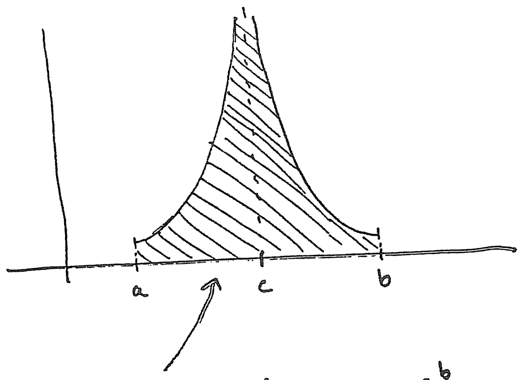
- the interval $[a, b]$ is finite
- the values of $f(x)$ are bounded.

We may be interested in areas over infinite intervals:

$$\int_a^\infty f(x)\,dx$$



or areas in regions where the function becomes unbounded:

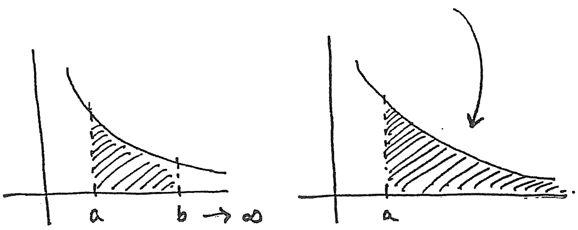


$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

> **Improper integrals** are defined to be limits of certain definite integrals, provided these limits exist.

Below we define the area to be

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$



> If the limit exists and is finite we say the improper integral **converges**.

> If the limit does not exist or is infinite we say the improper integral **diverges**.

Example:

$$\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2}$$

$$= \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b$$

$$= \lim_{b\to\infty} \left(-\frac{1}{b} - \left(-\frac{1}{1}\right)\right)$$

$$= \lim_{b\to\infty} -\frac{1}{b} + 1$$

$$= 0 + 1 = 1.$$

so this improper integral converges.

Example:

$$\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x}$$

$$= \lim_{b\to\infty} \Big[\ln x\Big]_1^b$$

$$= \lim_{b\to\infty} (\ln b - \ln 1)$$

$$= \lim_{b\to\infty} \ln b$$

$$= \infty$$

so this improper integral diverges.

Example:

$$\int_{-\infty}^0 e^x\,dx = \lim_{a\to-\infty} \int_a^0 e^x\,dx$$

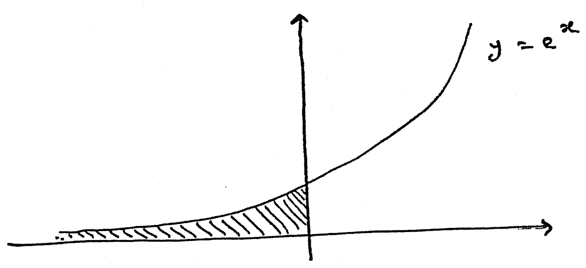
$$= \lim_{a\to-\infty} \Big[e^x\Big]_a^0$$

$$= \lim_{a\to-\infty} (e^0 - e^a)$$

$$= 1 - \lim_{a\to-\infty} e^a$$

$$= 1 - 0 = 1,$$

so the shaded area below is <u>1</u>:



Example:

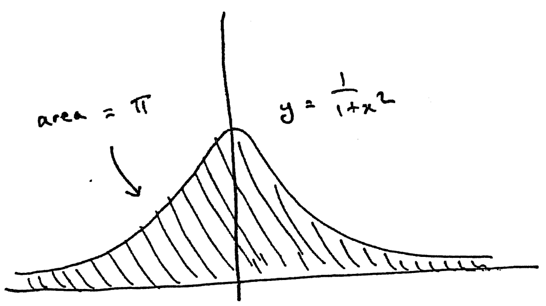
$$\int_{-\infty}^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_{-b}^b \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty} \Big[\tan^{-1} x\Big]_{-b}^b$$

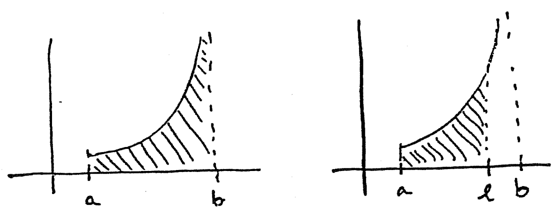
$$= \lim_{b\to\infty} \tan^{-1} b - \tan^{-1}(-b)$$

$$= 2 \lim_{b\to\infty} \tan^{-1} b$$

$$= 2\,\pi/2 = \pi$$



Other types of improper integrals occur when the integrand becomes unbounded:



In this illustration we define

$$\int_a^b f(x)\,dx = \underbrace{\lim_{l\to b^-}}_{} \int_a^l f(x)\,dx$$

note that this is a <u>one-sided</u> limit

Example :

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{\ell \to 1^-} \int_0^\ell \frac{dx}{\sqrt{1-x}}$$

$$= \lim_{\ell \to 1^-} \left[-2\sqrt{1-x}\right]_0^\ell$$

$$= \lim_{\ell \to 1^-} \left(-2\sqrt{1-\ell} + 2\right)$$

$$= 0 + 2 = 2 .$$

$$\int_1^2 \frac{dx}{1-x} = \lim_{\ell \to 1^+} \int_\ell^2 \frac{dx}{1-x}$$

$$= \lim_{\ell \to 1^+} \left[-\ln|1-x|\right]_\ell^2$$

$$= \lim_{\ell \to 1^+} -\ln|-1| + \ln|1-\ell|$$

$$= \lim_{\ell \to 1^+} \ln|1-\ell|$$

$$= -\infty \text{, so diverges.}$$

Example :

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}}$$

$$= \lim_{\ell \to 2^-} \int_1^\ell \frac{dx}{(x-2)^{2/3}} + \lim_{\ell \to 2^+} \int_\ell^4 \frac{dx}{(x-2)^{2/3}}$$

$$= \lim_{\ell \to 2^-} \left[3(x-2)^{1/3}\right]_1^\ell + \lim_{\ell \to 2^+} \left[3(x-2)^{1/3}\right]_\ell^4$$

$$= \lim_{\ell \to 2^-} \left(3(\ell-2)^{1/3} + 3\right)$$

$$+ \lim_{\ell \to 2^+} \left(3\sqrt[3]{2} - 3(\ell-2)^{1/3}\right)$$

$$= 3 + 3\sqrt[3]{2} .$$

Series and Taylor Polynomials

An infinite series (or just series) is an expression of the form

$$a_0 + a_1 + a_2 + \cdots + a_k + \cdots$$

which may be abbreviated to

$$\sum_{k=0}^{\infty} a_k \quad \text{or} \quad \sum a_k ,$$

and it represents

$$\lim_{n \to \infty} \sum_{k=0}^{n} a_k$$

$$= \lim_{n \to \infty} \left(a_0 + a_1 + \cdots + a_n\right)$$

Thus the series $\sum_{k=0}^{\infty} a_k$ is the limit of the sequence whose $(n+1)$th term is the partial sum

$$\sum_{k=0}^{n} a_k = a_0 + a_1 + \cdots + a_n .$$

If $\sum_{k=0}^{\infty} a_k$ exists and is finite, then we say the series converges.

If $\sum_{k=0}^{\infty} a_k$ does not exist, or is ∞ or $-\infty$, then we say the series diverges.

$$\boxed{\begin{array}{l}\underline{\text{Geometric series}}: \\ \displaystyle\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \dots \\ \qquad\qquad \dots + ar^k + \dots \\ \text{where } a, r \text{ are constants} \\ (r \text{ for "common ratio"}).\end{array}}$$

Put $S_n = a + ar + \dots + ar^n$

so $rS_n = ar + \dots + ar^n + ar^{n+1}$

$S_n - rS_n = a - ar^{n+1}$

so $(1-r)S_n = a(1-r^{n+1})$

so $\boxed{S_n = \dfrac{a(1-r^{n+1})}{1-r}}$

Hence

$$\sum ar^k = \lim_{n\to\infty} \sum_{k=0}^{n} ar^k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^{n+1})}{1-r}.$$

But

$$r^{n+1} \to \begin{cases} 0 & \text{if } |r|<1 \\ \infty & \text{if } r>1 \\ \text{undefined} & \text{if } r<-1. \end{cases}$$

Hence

$$\boxed{\sum ar^k = \frac{a}{1-r} \quad \text{if } |r|<1.}$$

$$\boxed{\text{If } |r| \geq 1 \text{ then the geometric series diverges.}}$$

e.g. $0.3333\dots$

$= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \dots$

$= \frac{a}{1-r}$ where $a = 3/10$, $r = \frac{1}{10}$

$= \frac{3/10}{1-1/10} = \frac{3/10}{9/10} = 1/3$

as expected !!

e.g. $1 - 1 + 1 - 1 + 1 - 1 + \dots$

is a geometric series where $a = 1$, $r = -1$ and diverges

(the partial sums are 1 and 0).

$$\boxed{\begin{array}{l}\text{The } \underline{\text{harmonic series}} \text{ is} \\ \displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \\ \text{which diverges:} \\ \boxed{\displaystyle\sum \frac{1}{k} = \infty}\end{array}}$$

Reason:

$$1 + \underbrace{\frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \dots$$
$$\dots + \underbrace{\frac{1}{2^{n-1}+1} + \dots + \frac{1}{2^n}}$$

$$\geq 1 + \underbrace{\frac{1}{2}} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}} + \dots$$
$$\dots + \underbrace{\frac{1}{2^n} + \dots + \frac{1}{2^n}}$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2}$$

$$= 1 + \frac{n}{2} \longrightarrow \infty \quad \text{as } n \to \infty.$$

Ratio Test for convergence:

Let $L = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right|$.

Then $\sum_{k=0}^{\infty} a_k$

(a) converges if $L < 1$;

(b) diverges if $L > 1$.

Note: if $L = 1$ then the Ratio Test tells us nothing.

Example: Consider the familiar geometric series:

$$a + ar + ar^2 + \dots + ar^k + \dots$$

$$= \sum_{k=0}^{\infty} a_k$$

where $a_k = ar^{k-1}$ for each k.

Then $\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right|$

$$= \lim_{k \to \infty} \left| \frac{ar^k}{ar^{k-1}} \right| = \lim_{k \to \infty} |r| = |r|$$

so the Ratio Test tells us (as we already know!) that

$$\text{geometric series} \begin{cases} \text{converges} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| > 1. \end{cases}$$

Example: $\sum_{k=0}^{\infty} \frac{1}{k!}$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{k!} + \dots$$

converges since

$$\lim_{k \to \infty} \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \lim_{k \to \infty} \frac{k!}{(k+1)!}$$

$$= \lim_{k \to \infty} \frac{1}{k+1} = 0 < 1.$$

[In fact, the series converges to e, see below.]

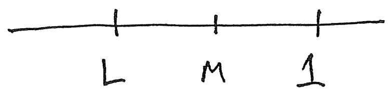
Reason for the Ratio Test:

We just consider (a) and suppose each term $a_k > 0$.

Suppose

$$\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = L < 1.$$

$\boxed{\text{Choose } M \text{ half-way between } L \text{ and } 1.}$



For large enough k, say $k \geq K$

$$\frac{a_{k+1}}{a_k} \leq M.$$

2.73

Hence, for $k \geq K$

$$\boxed{a_{k+1} \leq M a_k ,}$$

so

$$a_{K+1} \leq M a_K ,$$

$$a_{K+2} \leq M a_{K+1} \leq M^2 a_K ,$$

$$a_{K+3} \leq M a_{K+2} \leq M^3 a_K ,$$

and so on, so our series is

$$\underbrace{a_0 + a_1 + \cdots + a_{K-1}}_{\text{"junk"}} + \underbrace{a_K + a_{K+1} + a_{K+2} + \cdots}_{\text{"controlled"}}$$

$$\leq \text{"junk"} + a_K + M a_K + M^2 a_K + \cdots$$

$$= \text{"junk"} + a_K (1 + M + M^2 + \cdots)$$

2.74

But $M < 1$, so

$$\boxed{1 + M + M^2 + \cdots = \frac{1}{1-M} ,}$$

(convergent geometric series)

so our series is

$$\sum_{k=0}^{\infty} a_k \leq \text{"junk"} + a_K \frac{1}{1-M}$$

$$< \infty \quad !!!$$

Hence the partial sums $\sum_{k=0}^{n} a_k$ form an increasing bounded sequence.

By the Monotone Convergence Theorem, the partial sums form a convergent sequence, so

$$\boxed{\sum_{k=0}^{\infty} a_k \text{ converges.}}$$

2.75

Power series

Given a real number x,

a <u>power series in</u> x has the form

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

where $a_0, a_1, a_2, \ldots, a_n, \ldots$

are constants, and

x is a "variable".

(the word "power" comes from the use of <u>powers</u> of x in the terms of the series).

2.76

Think of a power series as an "infinite polynomial".

Convergence or divergence may vary according to choice of the real number x

e.g. the power series

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots + x^k + \cdots$$

is a geometric series which

$$\begin{cases} \text{converges} & \text{if } |x| < 1 \\ \text{diverges} & \text{if } |x| \geq 1 . \end{cases}$$

2.77

If we are very lucky, a given power series may converge for all x !!

Example : Let

$$P(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^k}{k!} + \dots$$

Observe that

$$\lim_{k\to\infty} \left| \frac{x^{k+1}/(k+1)!}{x^k/k!} \right| = \lim_{k\to\infty} |x| \frac{k!}{(k+1)!}$$

$$= \lim_{k\to\infty} \frac{|x|}{k+1} = 0 < 1 .$$

2.78

Hence, by the Ratio Test,

$P(x)$ converges for all x !!

Nice fact : Power series may be "differentiated" like ordinary polynomials.

Here

$$P'(x) = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \dots + \frac{kx^{k-1}}{k!} + \frac{(k+1)x^k}{(k+1)!} + \dots$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^{k-1}}{(k-1)!} + \frac{x^k}{k!} + \dots$$

$$= P(x) \quad !!!$$

2.79

Recall $\frac{d}{dx} e^x = e^x$.

In fact (in a sense to be made precise shortly)

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

for all x .

called a power series expansion of e^x.

2.80

Representing functions by power series.

A function $y = f(x)$ is represented by a power series $\sum_{k=0}^{\infty} a_k x^k$

if

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

whenever the R.H.S. converges, and the R.H.S. is called a power series expansion of $f(x)$.

– an "infinite polynomial" version of $f(x)$.

2.81

How do the constants a_k relate to the rule for f ?

Suppose everything is well-behaved, and we can differentiate as much as we like:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n + \dots$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1} + \dots$$

$$f''(x) = 2a_2 + (3)(2) a_3 x + \dots + n(n-1) a_n x^{n-2} + \dots$$

$$f'''(x) = (3)(2) a_3 + \dots + n(n-1)(n-2) a_n x^{n-3} + \dots$$

$$\vdots$$

$$\underbrace{f^{(n)}(x)}_{\text{nth derivative}} = \underbrace{n(n-1)(n-2)\dots(2)\, a_n}_{\text{constant term}} + \underbrace{\dots}_{\text{other "stuff"}}$$

2.82

Evaluating at $x = 0$ gives

$$f(0) = a_0$$

$$f'(0) = a_1$$

$$f''(0) = 2a_2$$

$$f'''(0) = (3)(2)\, a_3$$

$$\vdots$$

$$f^{(n)}(0) = n(n-1)\dots(3)(2)\, a_n .$$

Thus

$$a_0 = f(0)$$

$$a_1 = f'(0)$$

$$a_2 = \frac{f''(0)}{2!}$$

$$\vdots$$

$$\boxed{a_n = \frac{f^{(n)}(0)}{n!}} .$$

2.83

Thus we get the

Maclaurin series for $f(x)$:

$$\boxed{f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots}$$

Example: $f(x) = e^x$.

Observe $f^{(n)}(x) = e^x$ for all n,

so

$$a_n = \frac{e^0}{n!} = \frac{1}{n!}$$

and we get

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots}$$

2.84

Thus, for example,

$$e = 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!} + \dots$$

$$e^2 = 1 + 2 + \frac{2^2}{2!} + \dots + \frac{2^n}{n!} + \dots$$

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!} + \dots$$

Example: $f(x) = \sin x$.

$f(x) = \sin x$, $\quad f(0) = 0$,

$f'(x) = \cos x$, $\quad f'(0) = 1$,

$f''(x) = -\sin x$, $\quad f''(0) = 0$,

$f'''(x) = -\cos x$, $\quad f'''(0) = -1$,

$f^{(4)}(x) = \sin x$, $\quad f^{(4)}(0) = 0$,

then pattern reproduces forever ...

2.85

Thus the Maclaurin series for $\sin x$ is

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

Differentiating gives the Maclaurin series for $\cos x$:

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$$

What would happen if we made all the $-$ into $+$ in the series for $\sin x$ and $\cos x$

?!

2.86

Let

$$P(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$Q(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

Add these together:

$$\begin{aligned} P(x) + Q(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ &= e^x \quad !! \end{aligned}$$

Subtract $P(x)$ from $Q(x)$:

$$\begin{aligned} Q(x) - P(x) &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots \\ &= 1 - x + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \cdots \\ &= e^{-x} \quad !!! \end{aligned}$$

2.87

Thus

$$\begin{aligned} Q(x) + P(x) &= e^x \\ Q(x) - P(x) &= e^{-x} \\ \hline 2Q(x) &= e^x + e^{-x} \end{aligned}$$

so

$$\boxed{Q(x) = \frac{e^x + e^{-x}}{2} = \cosh x}$$

and

$$\begin{aligned} P(x) &= e^x - Q(x) \\ &= e^x - \frac{e^x + e^{-x}}{2} \\ &= \frac{e^x - e^{-x}}{2} \\ &= \sinh x . \end{aligned}$$

2.88

In summary,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

What about a power series representation of

$f(x) = \ln x$?

— no Maclaurin series because $f(0), f'(0), \ldots$ are undefined !!

More generally,

the <u>Taylor series expansion</u> or <u>representation</u> of

$$y = f(x)$$

<u>about</u> $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

The Maclaurin series

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

is the $\boxed{\text{Taylor series about } x = 0.}$

<u>Example</u>: Taylor series about $x = 1$ for $f(x) = \ln x$:

$$f(x) = \ln x, \quad f(1) = 0$$

$$f'(x) = 1/x, \quad f'(1) = 1$$

$$f''(x) = -1/x^2, \quad f''(1) = -1$$

$$f'''(x) = 2/x^3, \quad f'''(1) = 2$$

$$\vdots$$

$$\boxed{f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{x^n}, \quad f^{(n)}(1) = (-1)^{n-1}(n-1)!}$$

The Taylor series about $x = 1$ is

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots + (-1)^{n-1}\frac{(x-1)^n}{n} + \cdots$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(x-1)^k}{k}$$

When does this converge?

Applying the ratio test:

$$\lim_{k\to\infty}\left(\frac{|x-1|^{k+1}}{k+1}\,\frac{k}{|x-1|^k}\right)$$

$$= \lim_{k\to\infty}\left(|x-1|\frac{k}{k+1}\right) = |x-1|.$$

Thus the Taylor series

$$\begin{cases} \text{converges} & \text{if } 0 < x < 2 \\ \text{diverges} & \text{if } x < 0 \text{ or } x > 2. \end{cases}$$

If $x = 0$ then the series becomes

$$-1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{4} - \cdots - \tfrac{1}{n} - \cdots,$$

the negative harmonic series, which diverges.

If $x = 2$ it becomes

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots + (-1)^{n-1}\tfrac{1}{n} + \cdots$$

which converges, by a <u>tricky</u> application of the Monotone Convergence Theorem (fun exercise, try it !!!).

Another way to get the series for $\ln x$:

$$\boxed{\text{Exploit the fact that } \ln'(x) = \frac{1}{x}.}$$

$$\frac{1}{x} = \frac{1}{1-(1-x)}$$

$$= 1 + (1-x) + (1-x)^2 + (1-x)^3 + \cdots$$

(geometric series, $r = 1-x$)

so, antidifferentiating :

$$\ln x = x + \frac{(1-x)^2}{2}(-1) + \frac{(1-x)^3}{3}(-1) + \frac{(1-x)^4}{4}(-1) + \cdots + C$$

constant of integration

This gives

$$\boxed{\ln x = x - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + C.}$$

But

$$0 = \ln 1 = 1 - (0+0-0+\cdots) + C$$

so $\boxed{c = -1}$

yielding

$$\boxed{\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots,}$$

as before.

Taylor polynomials

> A Taylor series expansion is a limit of a sequence of polynomial approximations.

Define the Taylor polynomial of degree n for a function $y = f(x)$ about $x = a$ to be

$$\boxed{T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.}$$

> Thus the Taylor polynomial is obtained from the Taylor series by "chopping off" everything after the term involving $(x-a)^n$.

> When $a = 0$ the Taylor polynomial is also called the Maclaurin polynomial.

Example: The Taylor series about $x=0$ for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

Hence the Taylor (Maclaurin) polynomials of degree 0, 1, 2, 3 are

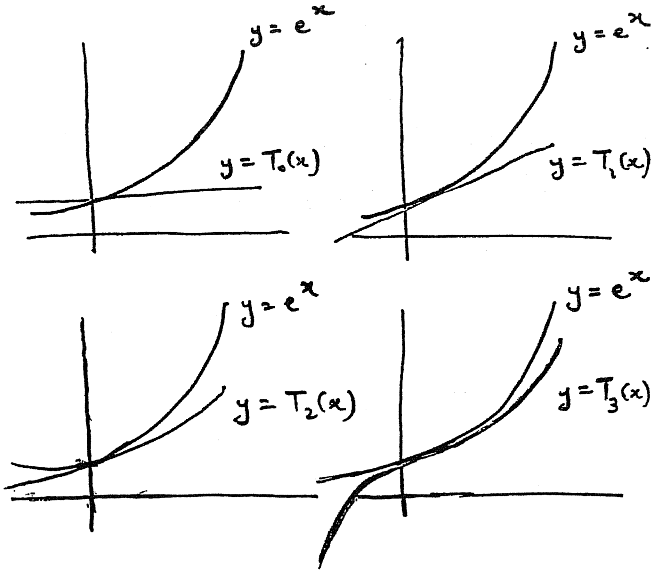
$$T_0(x) = 1$$

$$T_1(x) = 1 + x$$

$$T_2(x) = 1 + x + \frac{x^2}{2}$$

$$T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

As n increases, $y = T_n(x)$ "hugs" more of the graph of $y = f(x)$.



Example: The Taylor series about $x=0$ for $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The first few Taylor polynomials are

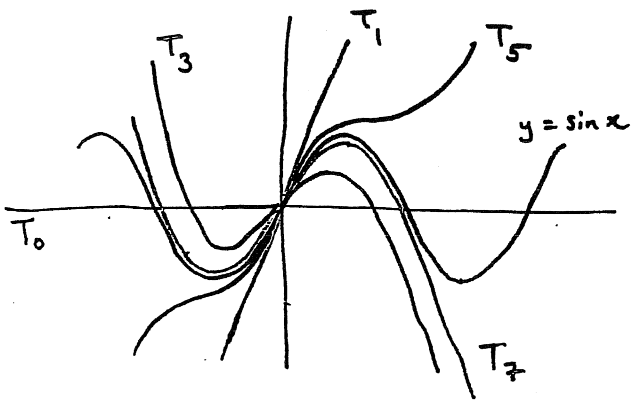
$$T_0(x) = 0,$$

$$T_1(x) = x = T_2(x),$$

$$T_3(x) = x - \frac{x^3}{6} = T_4(x),$$

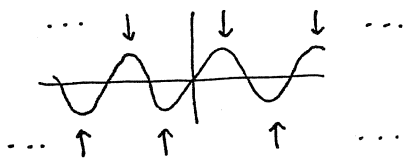
$$T_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120} = T_6(x)$$

$$T_7(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}.$$



$y = \sin x$ can never equal a polynomial in x

because its graph has infinitely many local extrema



whereas a polynomial has only finitely many critical points.

However as $n \to \infty$, the graph of $y = T_n(x)$ captures more and more of the sine's "wriggles".