# Notes on Integral Calculus and Modelling
## 2nd Instalment

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David Easdown, 17 July 2017
2.1

Logs and exponentials

Let $a, b > 0$.

Easy to understand:
- addition

\[ a \to a \to a \to a \to 0 \to a + b \to \]

translation along the real line

- multiplication

\[ \begin{array}{c}
\text{area of a} \\
\text{rectangle}
\end{array} \]

2.2

But how does one define or conceptualize exponentiation?!

Example: What is $a^{\frac{m}{n}}$?

\[ a^x = a \times a \times \ldots \times a \quad \text{if } n \in \mathbb{Z}^+ \]

\[ a^\frac{m}{n} = c \quad \text{where} \]

\[ x \times x \times \ldots \times x = a \quad \text{if } m \in \mathbb{Z}^+ \]

\[ a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = c \]

so we have $a^q$ for any $q \in \mathbb{Q}^+$

2.3

Easy fact: For all $q_1, q_2 \in \mathbb{Q}^+$

\[ q_1 < q_2 \Rightarrow a^{q_1} < a^{q_2} \]

Let $q_1 < q_2 < \ldots < q_n < \ldots < \Pi < 4$ where each $q_i \in \mathbb{Q}^+$ and

\[ \Pi = \lim_{n \to \infty} q_n \]

(e.g., use the decimal expansion of $\Pi$).

Then

\[ a^{q_1} < a^{q_2} < \ldots < a^{q_n} < \ldots < a^{\Pi} \]

monotonic sequence

so

\[ \lim_{n \to \infty} a^{q_n} \text{ exists} !! \]

2.4

Reason:

Monotone Convergence Theorem:

Let $x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_n \leq \ldots \leq M$

be an infinite non-decreasing sequence of real numbers bounded
above by $M$.

Then $\lim_{n \to \infty} x_n$ exists.

Idea: eventually the numbers "bunch up"

\[ x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_n \leq \ldots \leq M \]

limit
2.5

Proof: Put \( X = \{ x_n \mid n \in \mathbb{Z}^+ \} \).
Then \( X \) is bounded above by \( M \), so by completeness of \( \mathbb{R} \),
\( X \) has a least upper bound \( L \).

Completeness of \( \mathbb{R} \) says:
any nonempty set of reals which
is bounded above has a least
upper bound.

Claim: \( \lim_{n \to \infty} x_n = L \)

We have to prove
\((\forall \varepsilon > 0)(\exists N \in \mathbb{Z}^+)(\forall n \geq N)
| x_n - L | < \varepsilon \).

2.6

Let \( \varepsilon > 0 \).
If \( x_n \leq L - \varepsilon \) for all \( n \),
then \( L - \varepsilon \) is an upper bound
for \( X \), smaller than \( L \),
contradicting that \( L \) is the
least upper bound.

Hence \( L - \varepsilon < x_n \leq L \)
for some \( N \).

Then
\( L - \varepsilon < x_{N+1} \leq x_{N+2} \leq \ldots \leq L \)

so
\( | x_n - L | < \varepsilon \ \forall n \geq N \).

This proves
\( \lim_{n \to \infty} x_n = L \).

2.8

Completely different approach !!!

We will define
\[ a = b \ln a \]
provided we can make sense of

- \( \ln \) “the natural logarithm
of \( a \)”
- the real number \( e \)
- arbitrary powers of \( e \).

Advantage of this method:
“constructive” rather than
“existential”
2.9

<table>
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<td>$x^{3/4}$</td>
</tr>
<tr>
<td>$x^2$</td>
<td>$x^{3/2}$</td>
</tr>
<tr>
<td>$x$</td>
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<td>$x^0$</td>
<td>$x^{-1}$</td>
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<td>$\frac{1}{x}$</td>
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<td>$\frac{1}{x^3}$</td>
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<td>$x^{-n}$</td>
<td>$\frac{x^{n+1}}{n+1}$ if $n \neq -1$</td>
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By the Fundamental Theorem of Calculus (part 1)

$$\frac{d}{dx} \int_{1}^{x} \frac{1}{t} \, dt = \frac{1}{x}$$

Define the natural logarithmic function $\ln x$ by, for $x > 0$,

$$\ln x = \int_{1}^{x} \frac{1}{t} \, dt$$

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**Properties:**

| $\ln 1 = \int_{1}^{1} \frac{1}{t} \, dt = 0$ |
| If $x > 1$ then $\ln x = \int_{1}^{x} \frac{1}{t} \, dt > 0$ (positive area) |
| If $0 < x < 1$ then $\ln x = \int_{1}^{x} \frac{1}{t} \, dt$ $\Rightarrow -\int_{x}^{1} \frac{1}{t} \, dt < 0$ |

2.12

**The derivative:**

| If $x > 0$ then $\frac{d}{dx} \ln x = \frac{1}{x}.$ $^\star$ |
| If $x < 0$ then $-x > 0$ and $\frac{d}{dx} \ln(-x) = \frac{d}{du} \ln(u) \frac{du}{dx}$ where $u = -x$ $\Rightarrow \frac{1}{u} (-1)$ $\Rightarrow -\frac{1}{-x} = \frac{1}{x}.$ |

If $x > 0$ then $\frac{d}{dx} \ln(-x) = \frac{1}{x}.$ $^\star\star$

Combining ($\star$) and ($\star\star$):

| If $x \neq 0$ then $\frac{d}{dx} \ln|x| = \frac{1}{x}$ |
Logarithms "turn products into sums":  

For \( a, b > 0 \)  

\[ \ln(ab) = \ln(a) + \ln(b). \]

Proof: Fix \( a > 0 \) and define  

\[ g(x) = \ln(ax) \text{ for } x > 0. \]

Then  

\[ g'(x) = \frac{d}{dx} \ln(ax) = \frac{d}{du} \ln(u) \left( \frac{du}{dx} \right) = \frac{1}{u} \cdot a = \frac{a}{ax} = \frac{1}{x}. \]

\[ g(x) = \ln x + C \]

for some constant \( C \).

But  

\[ g(1) = \ln(a \cdot 1) = \ln a = \ln 1 + C = 0 + C = C, \]

so  

\[ C = \ln a. \]

Hence  

\[ g(x) = \ln x + \ln a. \]

In particular  

\[ \ln(ab) = g(b) = \ln b + \ln a = \ln a + \ln b, \]

as required.

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The exponential function

\[ \frac{d}{dx} \ln x = \frac{1}{x} > 0 \text{ for } x > 0 \]

so \( \ln x \) is an increasing function.

\[ \frac{d}{dx} \ln x = \frac{1}{x} \]

\[ y = x \]

\[ y = \ln x \]

Graph of inverse function

\[ y = x \]

\[ y = \ln x \]

Facts (tricky, proofs below)

\[ \lim_{x \to +\infty} \ln x = -\infty \]

\[ \lim_{x \to 0} \ln x = \infty \]

Hence

\[ g(x) = \ln x + C \]

for some constant \( C \).

But  

\[ g(1) = \ln(a \cdot 1) = \ln a = \ln 1 + C = 0 + C = C, \]

so  

\[ C = \ln a. \]

Hence  

\[ g(x) = \ln x + \ln a. \]

In particular  

\[ \ln(ab) = g(b) = \ln b + \ln a = \ln a + \ln b, \]

as required.

2.16

Using a lower Riemann sum we get

\[ \int_1^{2^n} \frac{dx}{x} \geq \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^n} \]

\[ = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{2^n} \]

\[ + \cdots + \frac{1}{2^{n-1}} + \cdots + \frac{1}{2^n} \]

\[ \geq \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{2^n} \]

\[ + \cdots + \frac{1}{2^n} + \cdots + \frac{1}{2^n} \]
so \[ \sum_{i=1}^{n} \frac{1}{i} > \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^n} \]

Thus
\[ \ln(2^n) \geq \frac{n}{2^n} \]

But
\[ \lim_{n \to \infty} \frac{n}{2^n} = 0 \]

so
\[ \lim_{n \to \infty} \ln(2^n) = 0 \]

(Squeeze Law)

so
\[ \lim_{n \to \infty} \ln(n) = -\infty \quad (n \in \mathbb{R}^+ \text{and } n \to \infty) \]

so
\[ \lim_{x \to \infty} \ln x = \infty \quad (x \in \mathbb{R}^+) \]

Corollary: \[ \lim_{x \to 0^+} \ln x = -\infty \]

Proof:
\[ \lim_{x \to 0^+} \ln x = \lim_{y \to \infty} \ln \left( \frac{1}{y} \right) \]

= \lim_{x \to 0} \ln \left( \frac{1}{x} \right) \]

= \lim_{x \to \infty} (-\ln x) \]

(Why?)

= -\lim_{x \to \infty} \ln x \]

= -\infty \]

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Thus
\[ \exp(\ln x) = x \quad \text{for } x > 0 \]

\[ \ln(\exp x) = x \quad \text{for } x \in \mathbb{R} \]

\[ \exp \text{ and } \ln \text{ "undo each other".} \]

Properties of \exp:
\[ \exp(0) = 1 \quad \text{(since } \ln 1 = 0) \]

\[ \exp(x) > 0 \quad \text{for all } x \in \mathbb{R} \]

\[ \exp(a+b) = [\exp(a)][\exp(b)] \]

-called the exponential law.
Proof of the exponential law:
Let \( a, b \in \mathbb{R} \) and put \( x = \exp(a) \), \( y = \exp(b) \).
Then \( a = \ln(x) \), \( b = \ln(y) \) and
\[
\exp(a+b) = \exp(\ln(x) + \ln(y)) = \exp(\ln(xy))
\]
by a property of \( \ln \)
\[= xy \]
since \( \exp \) "undoes" \( \ln \)
\[= \exp(a) \exp(b)\] ,
as required.

Most important property:

\[
\frac{d}{dx} \exp(x) = \exp(x)
\]

Proof:
\[
x = \ln(\exp(x))
\]
so
\[
1 = \frac{dx}{dx} = \frac{d}{dx} (\ln(\exp(x)))
\]
\[
= \frac{d}{du} (\ln(u)) \frac{du}{dx}
\]
where \( u = \exp(x) \)
\[
= \frac{1}{u} \frac{du}{dx},
\]
whence
\[
\frac{d}{dx} \exp(x) = \frac{du}{dx} = u = \exp(x),
\]
as required.

Recapping, we have the natural logarithm

\[
\ln x = \int_{1}^{x} \frac{1}{t} \, dt
\]
for \( x > 0 \).

Recall the exponential function \( \exp \) is the inverse of \( \ln \),
so, for \( a > 0 \), \( b \in \mathbb{R} \)
\[
b = \ln(a) \iff \exp(b) = a
\]
Important properties:

1. \( \ln(ab) = \ln a + \ln b \)
2. \( \exp(a+b) = \exp(a) \cdot \exp(b) \)
3. \( \frac{d}{dx} \ln|x| = \frac{1}{x} \)
4. \( \frac{d}{dx} \exp(x) = \exp(x) \)

Put \( e = \exp(1) \), so \( \ln e = 1 \)

We want
\[
\exp(x) = e^x
\]

yet to be defined

so that the variable \( x \)
is an exponent.

Observe, for \( n \) positive integer,

\[
\ln(a^n) = \ln(a \cdot a \cdot \ldots \cdot a) \quad \text{\( n \) times}
= \ln a + \ln a + \ldots + \ln a \quad \text{\( n \) times}
= n \ln(a).
\]

For \( n \) positive integer

\[
\ln(a^n) = n \ln(a)
\]

so

\[
a^n = \exp(n \ln(a))
\]

This suggests the following definition:

For \( x \in \mathbb{R} \) and \( a > 0 \) define

\[
a^x = \exp(x \ln(a))
\]

In particular

\[
e^x = \exp(x \ln(e)) = \exp(x)
\]

Properties:

1. \( (a^b)^c = a^{bc} \)
2. \( a^c \cdot a^d = a^{c+d} \)
3. \( (a^c)^d = a^{cd} \)
4. \( a^0 = 1 \)
5. \( a^n = a \times a \times \ldots \times a \quad \text{\( n \) times} \quad \text{if \( x \in \mathbb{Z}^+ \)}
6. \( \ln(a^x) = x \ln(a) \)
7. \( \frac{d}{dx} a^x = a^x \ln a \)

Proofs: left as exercises.
Further techniques of integration

Integration by parts:

Recall the product rule

\[
\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}
\]

where \( u, v \) functions of \( x \).

Antidifferentiate both sides with respect to \( x \):

\[
\int \left[ u \frac{dv}{dx} \right] dx = \int \left[ \frac{d}{dx}(uv) \right] dx
\]

\[
= \int u \frac{dv}{dx} \, dx + \int v \frac{du}{dx} \, dx
\]

\[
= \int u \, dv + \int v \, du
\]

Rearranging yields the integration by parts formula:

\[
\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx
\]

or, more simply,

\[
\int u \, dv = uv - \int v \, du
\]

Example:

\[
\int xe^x \, dx = xe^x - \int e^1 \, dx
\]

\[
= xe^x - \int e^x \, dx
\]

\[
= xe^x - e^x + C
\]

Trying \( u = e^x \), \( \frac{dv}{dx} = x \)

would make things more complicated!
Example:
\[ \int x^2 e^x \, dx = \int x e^x - \int e^x (2x) \, dx \]
\[ = \int x e^x - 2 \int x e^x \, dx \]
\[ = e^x \left( x^2 - 2x + 2 \right) + C \]

Handy trick: Sometimes \( \frac{dv}{dx} = 1 \) helps.

Example:
\[ \int \ln x \, dx = \int 1 \cdot \ln x \, dx \]
\[ = \int \frac{du}{u} \]
\[ = \ln x - \int x \left( \frac{1}{x} \right) \, dx \]
\[ = \ln x - \int 1 \, dx \]
\[ = \ln x - x + C \]

Another trick: use parts to express an integral in terms of itself and rearrange.

Example:
\[ \int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx \]
\[ = e^x \sin x - \int e^x \cos x \, dx \]

Put \( I = \int e^x \sin x \, dx \)

\[ I = e^x \sin x - \int e^x \cos x \, dx \]
\[ \Rightarrow 2I = e^x \sin x - e^x \cos x + C \]
\[ \Rightarrow I = \frac{e^x}{2} (\sin x - \cos x) + C \]
For definite integrals, use
\[ \int_a^b u \frac{dv}{dx} \, dx = [uv]_a^b - \int_a^b v \frac{du}{dx} \, dx \]

Example:
\[
\int e^x \ln x \, dx = \left[ \frac{e^x \ln x}{2} \right] - \int \frac{e^x}{x} \, dx
\]
\[ \frac{du}{dx} = u \quad \frac{dv}{dx} = e^x \ln x \]
\[ v = \frac{e^x \ln x}{2} \]
\[ u = \frac{e^x}{2} \]
\[ \int e^x \ln x \, dx = \frac{e^x \ln x}{2} - \frac{e^x}{2} \int e^x \, dx \]
\[ = \frac{e^x}{2} \ln x - \frac{1}{2} \int e^x \, dx \]
\[ = \frac{e^x}{2} \ln x - \frac{1}{2} \left( e^x - \frac{1}{2} \right) \]
\[ = \frac{e^x}{2} \ln x + \frac{1}{4} \]

Example: Develop a formula for \( \int \sin^n x \, dx \)

Solution:
\[
\int \sin^n x \, dx = \int \sin^{n-1} x \sin x \, dx
\]
\[ \frac{du}{dx} = u \quad \frac{dv}{dx} = \sin x \]
\[ v = \sin^{n-1} x \]
\[ u = \sin x \]
\[ \int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos x \, dx \]

Reduction formulae:
There are recursive formulae, allowing calculation in several steps - typically by reducing powers in an integrand.

Commonly, reduction formulae are derived using integration by parts.

Put \( I_n = \int \sin^n x \, dx \)
Then
\[ I_n = -\sin^{n-1} x \cos x \]
\[ + (n-1) \left[ I_{n-2} - I_n \right] \]
\[ = -\sin^{n-1} x \cos x \]
\[ + (n-1) I_{n-2} - (n-1) I_n \]

Hence
\[ n I_n = I_n + (n-1) I_n \]
\[ = -\sin^{n-1} x \cos x + (n-1) I_{n-2} \]

whence
\[ I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2} \]

**Partial fractions and rational functions**

A rational function is a quotient (ratio)

of polynomials:

\[ f(x) = \frac{P(x)}{Q(x)} \]

where \( P(x), Q(x) \) are polynomials.

If \( \frac{P(x)}{Q(x)} \) is a constant

\[ \int P(x) \, dx = (ax + b)^m \]

or \( \frac{P(x)}{Q(x)} \) is linear

\[ \int P(x) \, dx = (ax^2 + bx + c)^m \]

then it is possible to antidifferentiate using techniques so far discussed.

Otherwise, we use the method of partial fractions to "decompose" the rational function into pieces which are of this form.

**Fundamental Theorem of Algebra**

Every polynomial \( p(x) \) with coefficients from

\[ \mathbb{C} = \{ \text{complex numbers} \} \]

can be factorized into linear factors

\[ p(x) = (x - \lambda_1)(x - \lambda_2) \ldots (x - \lambda_n) \]

for some \( \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C} \)

called roots.

If the coefficients of \( p(x) \) come from \( \mathbb{R} \), then the roots come in complex conjugate pairs

\[ \lambda_1 = a + ib, \quad \lambda_2 = a - ib, \ldots \]

If \( b \neq 0 \) then we get an irreducible quadratic factor:

\[ (x - \lambda_1)(x - \lambda_2) = x^2 - 2ax + a^2 + b^2 \]

Consequence: all real polynomials factorize into linear and irreducible quadratic factors.

This leads to the following method:
Method for decomposing \( \frac{P(x)}{Q(x)} \):

1. Divide through by \( Q(x) \) if the degree of \( P(x) \) is greater than or equal to the degree of \( Q(x) \).

2. Factorize \( Q(x) \) into linear and irreducible quadratic factors.

3. If \((x-a)\) is a factor, include a term \( \frac{A}{x-a} \).

4. If \((x-a)^n\) is a repeated factor, include terms \( \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n} \).

5. If \(x^2 + bx + c\) is an irreducible quadratic factor, include a term \( \frac{Ax + B}{x^2 + bx + c} \).

6. Analogous to (4) if \((x^2 + bx + c)^n\) is a repeated factor.

To find all constants that arise in (3), (4), (5), (6), put everything over a common denominator and equate numerators.

Either

(i) comparing coefficients of powers of \( x \)

or

(ii) using convenient substitutions for \( x \), enables constants to be found.

Example: Find

\[ \int \frac{dx}{(x-1)(x-2)(x-3)} \]

Solution: We find \( A, B, C \) such that

\[ \frac{1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} \]

giving

\[ 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \]

This must hold for all \( x \), by continuity of polynomials!!!

Judicious choices of \( x \) yield \( A, B, C \) quickly.
Put $x = 1$: $1 = A(-1)(-1)$, so $A = \frac{1}{2}$.
$x = 2$: $1 = B(1)(-1)$, so $B = -1$.
$x = 3$: $1 = C(3)(1)$, so $C = \frac{1}{2}$.

Thus
\[
\frac{1}{(x-1)(x-2)(x-3)} = \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{x-2}}{x-1} + \frac{\frac{1}{x-3}}{x-1}
\]

so
\[
\int \frac{dx}{(x-1)(x-2)(x-3)} = \frac{1}{2} \int \frac{dx}{x-1} - \frac{1}{x-2} + \frac{1}{x-3} + C
\]

\[= \frac{1}{2} \ln |x-1| - \ln |x-2| + \frac{1}{2} \ln |x-3| + C
\]

\[= \ln \left(\frac{1}{x-1}\right) + C
\]

---

Example: Find $\int \frac{dx}{x(x-1)^2}$.

Solution: Put
\[
\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}
\]

so
\[1 = A(x-1)^2 + Bx(x-1) + Cx.
\]

Put $x = 0$: $1 = A(0)^2$, so $A = 1$.
$x = 1$: $1 = C(1)$, so $C = 1$.
$x = 2$: $1 = A + 2B + 2C$, so $B = -1$.

Hence
\[\int \frac{dx}{x(x-1)^2} = \int \frac{dx}{x-1} - \int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2}
\]

\[= \ln |x-1| - \ln |x-1| + \frac{1}{x-1} + C
\]

\[= \ln |x-1| - \frac{1}{x-1} + C
\]

---

Example: Find $\int \frac{x^4 + x - 1}{x^3 + x} \, dx$.

Solution:
\[x^3 + x \int \frac{x^4 + x - 1}{x^3 + x} = x + \frac{-x^2 + x - 1}{x^3 + x}.
\]

So
\[\frac{x^4 + x - 1}{x^3 + x} = x + \frac{-x^2 + x - 1}{x^3 + x}.
\]

Put $-x^2 + x - 1 = A(x^2 + 1) + (Bx + C)x$.

Put $x = 0$: $-1 = A(0)$, so $A = -1$ giving
\[-x^2 + x - 1 = -x^2 + Bx + Cx.
\]

So
\[x = Bx^2 + Cx.
\]

---

Equating coefficients gives
\[B = 0, \ C = 1.
\]

Hence
\[\int \frac{x^4 + x - 1}{x^3 + x} \, dx = \int x \, dx - \int \frac{dx}{x^3 + x} + \int \frac{dx}{x^2 + 1}
\]

\[= \frac{x^2}{2} - \ln|x| + \tan^{-1} x + C
\]
Improper integrals

The definite integral
\[ \int_a^b f(x) \, dx \]
was developed assuming
- the interval \([a, b]\) is finite
- the values of \(f(x)\) are bounded

We may be interested in areas over infinite intervals:

\[ \int_a^\infty f(x) \, dx \]

or areas in regions where the function becomes unbounded:

\[ \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \]

Improper integrals are defined to be limits of certain definite integrals, provided these limits exist.

Below we define the area to be

\[ \int_a^\infty f(x) \, dx = \lim_{b \to \infty} \int_a^b f(x) \, dx \]

If the limit exists and is finite, we say the improper integral converges.

If the limit does not exist or is infinite, we say the improper integral diverges.

Example:

\[ \int_1^\infty \frac{dx}{x^2} = \lim_{b \to \infty} \int_1^b \frac{dx}{x^2} \]

\[ = \lim_{b \to \infty} \left[ -\frac{1}{x} \right]_1^b \]

\[ = \lim_{b \to \infty} \left( -\frac{1}{b} - (-1) \right) \]

\[ = \lim_{b \to \infty} -\frac{1}{b} + 1 \]

\[ = 0 + 1 = 1 \]

So this improper integral converges.
Example:
\[
\int_{-\infty}^{0} e^x \, dx = \lim_{a \to -\infty} \int_{a}^{0} e^x \, dx
\]
\[
= \lim_{a \to -\infty} \left[ e^x \right]_{a}^{0}
\]
\[
= \lim_{a \to -\infty} (e^0 - e^a)
\]
\[
= 1 - \lim_{a \to -\infty} e^a
\]
\[
= 1 - 0 = 1,
\]
so the shaded area below is 1:

\[ y = e^x \]

Example:
\[
\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \lim_{b \to \infty} \int_{-b}^{b} \frac{dx}{1 + x^2}
\]
\[
= \lim_{b \to \infty} \left[ \tan^{-1} x \right]_{-b}^{b}
\]
\[
= \lim_{b \to \infty} \tan^{-1} b - \tan^{-1} (-b)
\]
\[
= 2 \lim_{b \to \infty} \tan^{-1} b
\]
\[
= 2 \cdot \frac{\pi}{2} = \pi
\]

Other types of improper integrals occur when the integrand becomes unbounded:

In this illustration we define
\[
\int_{a}^{b} f(x) \, dx = \lim_{c \to b-} \int_{a}^{c} f(x) \, dx
\]

Note that this is a one-sided limit.
Example:

\[
\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{L \to 1^-} \int_0^L \frac{dx}{\sqrt{1-x}} \\
= \lim_{L \to 1^-} \left[ -2\sqrt{1-x} \right]_0^L \\
= \lim_{L \to 1^-} (-2\sqrt{1-L} + 2) \\
= 0 + 2 = 2.
\]

\[
\int_1^2 \frac{dx}{1-x} = \lim_{L \to 2^-} \int_1^L \frac{dx}{1-x} \\
= \lim_{L \to 2^-} \left[ -\ln |1-x| \right]_1^L \\
= \lim_{L \to 2^-} -\ln |1-L| + \ln |1-1| \\
= \lim_{L \to 1^+} \ln |1-L| \\
= -\infty, \text{ so diverged.}
\]

Series and Taylor Polynomials

An infinite series (or just series) is an expression of the form

\[ a_0 + a_1 + a_2 + \ldots + a_n + \ldots \]

which may be abbreviated to

\[ \sum_{k=0}^{\infty} a_k \]

and it represents

\[ \lim_{n \to \infty} \sum_{k=0}^{n} a_k. \]

Thus the series \( \sum_{k=0}^{\infty} a_k \) is the limit of the sequence whose \((n+1)\)th term is the partial sum

\[ s_n = a_0 + a_1 + \ldots + a_n. \]

If \( \sum_{k=0}^{\infty} a_k \) exists and is finite, then we say the series converges.

If \( \sum_{k=0}^{\infty} a_k \) does not exist, or is \( \infty \) or \( -\infty \), then we say the series diverges.
Geometric series:

\[ \sum_{k=0}^{\infty} a r^k = a + ar + ar^2 + \ldots + ar^k + \ldots \]

where \( a, r \) are constants

( \( r \) for "common ratio"").

Put \( S_n = a + ar + \ldots + ar^n \)

so \( \begin{align*}
S_n &= a + ar + \ldots + ar^n \\
r \cdot S_n &= \quad ar + \ldots + ar^n + ar^{n+1} \\
S_n - r \cdot S_n &= a - ar^{n+1} \\
S_n &= \frac{a(1 - r^{n+1})}{1 - r}
\end{align*} \)

Hence

\[ \sum_{k=0}^{\infty} a r^k = \lim_{n \to \infty} S_n = \frac{a}{1 - r} \]

But \( r^{n+1} \to \)

\[ \begin{cases} 
0 & \text{if } |r| < 1 \\
\infty & \text{if } r > 1 \\
\text{undefined} & \text{if } r < -1.
\end{cases} \]

If \( |r| \geq 1 \) then the geometric series diverges.

\[ \begin{align*}
\frac{3}{10} &= 0.33333 \ldots \\
&= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \ldots \\
&= \frac{a}{1 - r} \\
&= \frac{\frac{3}{10}}{1 - \frac{1}{10}} = \frac{3}{9} = \frac{1}{3}
\end{align*} \]

as expected!

\[ \begin{align*}
1 + 1 + 1 + 1 + 1 + \ldots
\end{align*} \]

is a geometric series where \( a = 1 \),

\( r = -1 \) and diverges

(the partial sums are 1 and 0).

The harmonic series is

\[ \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \]

which diverges:

\[ \sum_{k=1}^{\infty} \frac{1}{k} = \infty \]

Reason:

\[ \begin{align*}
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \ldots
\end{align*} \]

\[ \begin{align*}
&\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \ldots \\
&= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n}
\end{align*} \]

\[ \begin{align*}
&= 1 + \frac{\frac{1}{2}}{1 - \frac{1}{2}} = \infty \text{ as } n \to \infty.
\end{align*} \]
Ratio Test for convergence:

Let \( L = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| \).

Then \( \leq a_k \) for each \( k \).

(a) converges if \( L < 1 \);

(b) diverges if \( L > 1 \).

Note: if \( L = 1 \) then the Ratio Test tells us nothing.

Example:

\[
\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{k!} + \ldots
\]

converges since

\[
\lim_{k \to \infty} \left| \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} \right| = \lim_{k \to \infty} \frac{k!}{(k+1)!} = \lim_{k \to \infty} \frac{1}{k+1} = 0 < 1
\]

"In fact, the series converges to \( e \), see below."

Example: Consider the familiar geometric series:

\[
a + ar + ar^2 + \ldots + ar^k + \ldots
\]

where \( a_k = ar^{k-1} \) for each \( k \).

Then \( \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} |r| = |r| \)

so the Ratio Test tells us (as we already know!) that

a geometric series \( \sum a_k \) converges if \( |r| < 1 \)

and diverges if \( |r| > 1 \).

Reason for the Ratio Test:

We just consider (a) and suppose each term \( a_k > 0 \).

Suppose

\[
\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = L < 1
\]

Choose \( M \) halfway between \( L \) and 1.

\[
L \quad M \quad 1
\]

For large enough \( k \), say \( k \geq K \)

\[
\frac{a_{k+1}}{a_k} \leq M
\]
Hence, for \( k \geq K \)
\[
a_{k+1} \leq M a_k,
\]
so
\[
a_{k+2} \leq M a_{k+1} \leq M^2 a_k,
\]
and so on, so our series is
\[
a_0 + a_1 + \ldots + a_{k-1} + a_k + a_{k+1} + a_{k+2} + \ldots
\]
"junk" \[\Rightarrow\] "controlled"
\[
\leq "junk" + a_k + M a_k + M^2 a_k + \ldots
\]
\[
= "junk" + a_k (1 + M + M^2 + \ldots)
\]

But \( M < 1 \), so
\[
1 + M + M^2 + \ldots = \frac{1}{1-M},
\]
(convergent geometric series)
so our series is
\[
\sum_{k=0}^{\infty} a_k \leq "junk" + a_k \frac{1}{1-M}
\]
\[
< \infty \text{!!!}
\]

Hence the partial sums \( \sum_{k=0}^{\infty} a_k \)
form an increasing bounded sequence.

By the Monotone Convergence Theorem, the partial sums form a convergent sequence, so
\[
\sum_{k=0}^{\infty} a_k \text{ converges.}
\]

### Power series

Given a real number \( x \), a power series in \( x \) has the form
\[
\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k + \ldots
\]
where \( a_0, a_1, a_2, \ldots, a_n, \ldots \) are constants, and
\( x \) is a "variable".

( The word "power" comes from the use of powers of \( x \) in the terms of the series. )

### Convergence or divergence

Think of a power series as an "infinite polynomial".

Convergence or divergence may vary according to choice of the real number \( x \).

E.g., the power series
\[
\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \ldots + x^k + \ldots
\]
is a geometric series which
\[
\begin{cases}
\text{converges if } |x| < 1 \\
\text{diverges if } |x| \geq 1
\end{cases}
\]
If we are very lucky, a given power series may converge for all \( x \).!!

**Example:** Let

\[ p(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \]

\[ = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots \]

Observe that

\[ \lim_{k \to \infty} \frac{x^{k+1}/(k+1)!}{x^k/k!} = \lim_{k \to \infty} \frac{x^1}{k!} = 0 < 1 \]

Hence, by the Ratio Test,

\[ p(x) \] converges for all \( x \).!!

**Nice fact:** Power series may be "differentiated" like ordinary polynomials.

Here

\[ p'(x) = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots + \frac{k \cdot x^{k-1}}{(k-1)!} + \frac{(k+1)x^k}{k!} + \cdots \]

\[ = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{k-1}}{(k-1)!} + \frac{x^k}{k!} + \cdots \]

\[ = -p(x) \] !!

Recall \( \frac{d}{dx} e^x = e^x \).

In fact (in a sense to be made precise shortly)

\[ e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \]

\[ \uparrow \]

called a power series expansion of \( e^x \).

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Representing functions by power series.

A function \( y = f(x) \) is represented by a power series

\[ \sum_{k=0}^{\infty} a_k x^k \]

if

\[ f(x) = \sum_{k=0}^{\infty} a_k x^k \]

whenever the R.H.S. converges, and the R.H.S. is called a power series expansion of \( f(x) \).

- an "infinite polynomial" version of \( f(x) \).
How do the constants $a_n$ relate to the rule for $f$?

Suppose everything is well-behaved, and we can differentiate as much as we like:

\[
\begin{align*}
  f(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots \\
  f'(x) &= a_1 + 2a_2x + 3a_3x^2 + \cdots + n a_n x^{n-1} + \cdots \\
  f''(x) &= 2a_2 + (3)(2)a_3x + \cdots + n(n-1)a_n x^{n-2} + \cdots \\
  f^{(n)}(x) &= (3)(2)(1)a_3x^2 + \cdots + n(n-1)(n-2)a_n x^{n-3} + \cdots \\
  \vdots \\
  f^{(n)}(x) &= n(n-1)\cdots(2)a_n x + \cdots \\
\end{align*}
\]

Thus:

\[
\begin{align*}
  a_0 &= f(0) \\
  a_1 &= f'(0) \\
  a_2 &= \frac{f''(0)}{2!} \\
  \vdots \\
  a_n &= \frac{f^{(n)}(0)}{n!} \\
\end{align*}
\]

Thus we get the Maclaurin series for $f(x)$:

\[
\begin{align*}
  f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\
\end{align*}
\]

Example: $f(x) = e^x$.

Observe $f^{(n)}(x) = e^x$ for all $n$, so

\[
a_n = \frac{e^0}{n!} = \frac{1}{n!}
\]

and we get

\[
e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots
\]

Evaluating at $x = 0$ gives

\[
\begin{align*}
f(0) &= a_0 \\
f'(0) &= a_1 \\
f''(0) &= 2a_2 \\
f^{(n)}(0) &= n(n-1)\cdots(3)(2)a_n \\
\end{align*}
\]

Thus:

\[
\begin{align*}
a_0 &= f(0) \\
a_1 &= f'(0) \\
a_2 &= \frac{f''(0)}{2!} \\
\vdots \\
a_n &= \frac{f^{(n)}(0)}{n!} \\
\end{align*}
\]

Thus, for example,

\[
\begin{align*}
e &= 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots \\
e^x &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \\
e^{-1} &= 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(1)^n}{n!} + \cdots
\end{align*}
\]

Example: $f(x) = \sin x$.

\[
\begin{align*}
f(0) &= 0 \\
f'(0) &= 1 \\
f''(0) &= 0 \\
f^{(n)}(0) &= (-1)^n \\
f^{(n)}(x) &= \sin x \\
f^{(n)}(0) &= 0
\end{align*}
\]

then pattern reproduces forever...
Thus the Maclaurin series for $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

Differentiating gives the Maclaurin series for $\cos x$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

What would happen if we made all the $-$ into $+$ in the series for $\sin x$ and $\cos x$?!

Let

$$P(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$$

$$Q(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$$

Add these together:

$$P(x) + Q(x) = 1 + x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

$$= e^x ! !$$

Subtract $P(x)$ from $Q(x)$:

$$Q(x) - P(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$= 1 - x + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \ldots$$

$$= e^{-x} ! ! !$$

Thus

$$Q(x) + P(x) = e^x$$

$$Q(x) - P(x) = e^{-x}$$

Subtract $P(x)$ from $Q(x)$:

$$Q(x) = e^x + e^{-x}$$

So

$$Q(x) = \frac{e^x + e^{-x}}{2} = \cosh x$$

and

$$P(x) = e^x - Q(x)$$

$$= e^x - \frac{e^x + e^{-x}}{2}$$

$$= e^x - \frac{e^x - e^{-x}}{2}$$

$$= \frac{e^x - e^{-x}}{2} = \sinh x$$

In summary,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

What about a power series representation of $f(x) = \ln x$?!

- no Maclaurin series because $f(c), f'(c), \ldots$ are undefined !!
More generally,

the Taylor series expansion

or representation of

\[ y = f(x) \]

about \( x = a \) is

\[ f(x) = f(a) + f'(a)(x-a) \]

\[ + \frac{f''(a)}{2!} (x-a)^2 + \cdots \]

\[ + \frac{f^{(n)}(a)}{n!} (x-a)^n + \cdots \]

\[ = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k \]

The Macharini series

\[ f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} x^k \]

is the Taylor series about \( x = 0 \).

Example: Taylor series about \( x = 1 \) for \( f(x) = \ln x \):

\[ f(x) = \ln x, \quad f(1) = 0 \]

\[ f'(x) = \frac{1}{x}, \quad f'(1) = 1 \]

\[ f''(x) = -\frac{1}{x^2}, \quad f''(1) = -1 \]

\[ f'''(x) = 2\frac{1}{x^3}, \quad f'''(1) = 2 \]

\[ f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{x^n}, \quad f^{(n)}(1) = (-1)^{n-1} (n-1)! \]

The Taylor series about \( x = 1 \) is

\[ \ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} \]

\[ + \cdots + (-1)^{n-1} \frac{(x-1)^n}{n} + \cdots \]

\[ = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (x-1)^k}{k} \]

When does this converge?

Applying the ratio test:

\[ \lim_{k \to \infty} \left| \frac{1}{x-1} \right| | \frac{k}{k+1} \right| \]

\[ = \lim_{k \to \infty} \left| (x-1) \frac{k}{k+1} \right| = |x-1| \]

Thus the Taylor series

\[ \left\{ \begin{array}{ll}
\text{converges if } 0 < x < 2 \\
\text{diverges if } x < 0 \text{ or } x > 2
\end{array} \right. \]

If \( x = 0 \) then the series becomes

\[ -1 - \frac{1}{2} - \frac{1}{4} - \cdots - \frac{1}{n} - \cdots, \]

the negative harmonic series, which diverges.

If \( x = 2 \) it becomes

\[ 1 - \frac{1}{2} + \frac{1}{4} + \cdots + (-1)^{n-1} \frac{1}{2^n} + \cdots \]

which converges, by a tricky application of the Monotone Convergence Theorem (fun exercise, try it!!!).
Another way to get the series for \( \ln x \):

Exploit the fact that \( \ln'(x) = \frac{1}{x} \).

\[
\frac{1}{x} = \frac{1}{1 - (1 - x)} = 1 + (1 - x) + (1 - x)^2 + (1 - x)^3 + \ldots \\
\text{geometric series, } r = 1 - x
\]

so, anti-differentiating:

\[
\ln x = x + \frac{(1 - x)^2}{2} - (1 - x)^3 + \frac{(1 - x)^4}{4} + \ldots \\
\text{constant of integration}
\]

This gives

\[
\ln x = x - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots + C.
\]

Thus, \( C = -1 \)

yielding

\[
\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots
\]

as before.

---

**Taylor polynomials**

A Taylor series expansion is a limit of a sequence of polynomial approximations.

Define the Taylor polynomial of degree \( n \) for a function \( y = f(x) \) about \( x = a \) to be

\[
T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.
\]

Thus the Taylor polynomial is obtained from the Taylor series by "chopping off" everything after the term involving \( (x-a)^n \).

When \( a = 0 \) the Taylor polynomial is also called the Maclaurin polynomial.
Example: The Taylor series about $x=0$ for $e^x$ is
\[ e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \]
Hence the Taylor (Maclaurin) polynomials of degree 0, 1, 2, 3 are
\[ T_0(x) = 1 \]
\[ T_1(x) = 1 + x \]
\[ T_2(x) = 1 + x + \frac{x^2}{2!} \]
\[ T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{6} \]
As $n$ increases, $y = T_n(x)$ "hugs" more of the graph of $y = f(x)$.

The first four Taylor polynomials are
\[ T_0(x) = 0, \]
\[ T_1(x) = x = T_1(x), \]
\[ T_2(x) = x - \frac{x^3}{6} = T_2(x), \]
\[ T_3(x) = x - \frac{x^3}{6} + \frac{x^5}{120} = T_3(x) \]
\[ T_4(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} \]

Example: The Taylor series about $x=0$ for $\sin x$ is
\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \]

$y = \sin x$ can never equal a polynomial in $x$ because its graph has infinitely many local extrema

\[ \begin{array}{ccc}
\downarrow & \downarrow & \downarrow \\
\uparrow & \uparrow & \uparrow \\
\end{array} \]

whereas a polynomial has only finitely many critical points.

However as $n \to \infty$, the graph of $y = T_n(x)$ captures more and more of the sine's "wriggles".