The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial for Week 8

MATH1903: Integral Calculus and Modelling (Advanced) — Semester 2, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1903/
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Material covered

- ☐ Explicit first order differential equations for y only depending on y'
- ☐ Separation of variables
- ☐ Direction fields
- ☐ Aspects of modelling

Outcomes

After completing this tutorial you should

- ☐ be able to solve the simplest differential equations
- ☐ be able to solve equations by separation of variables
- ☐ be able to sketch direction fields and corresponding solutions of simple differential equations.
- ☐ be able to determine the asymptotic behaviour of solutions by looking at the explicit solution of a differential equation

Summary of essential material

What is a differential equation? A differential equation is an equation, where the unknown is a function. The equation involves that unknown function and some first or higher order derivatives of the unknown function.

Simplest differential equations: Explicit equation

$$\frac{dy}{dx}(x) = g(x) \qquad \text{with solution} \qquad y(x) = \int g(x)\,dx + C \quad \text{(anti-derivative of } g\text{)}.$$

The constant C is determined by an initial condition, for instance $y(0) = a$.

Separable differential equations: These are equations that can be written in the form

$$\frac{dy}{dx} = g\big(y(x)\big)h(x),$$

that is, the derivative y' can be written as a product of a function of y and a function of x. To solve we *separate variables*, putting all x on one side and all y on the other side:

$$\frac{dx}{h(x)} = g(y)dy, \qquad \text{then integrate:} \qquad \int \frac{dx}{h(x)} = \int g(y)dy,$$

and finally solve for y. This also involves an integration constant that is determined by an initial condition. Alternatively one can do a *definite integral* and build initial conditions directly into the calculation.

Direction fields: Assume that $y(x)$ is a solution of the differential equation $y'(x) = f\big(x, y(x)\big)$. The means that the graph $y = y(x)$ has slope $f\big(x, y(x)\big)$. Hence if we plot the slopes $f(x, y)$ at every point (x, y) in the relevant region of the plane, all solutions of the differential equation are tangent to the direction field. We often proceed as follows to sketch the direction field:

- Find *stationary points*, that is, points (x, y) such that $y'(x) = f(x, y) = 0$. If y_0 is such that $f(x, y_0) = 0$ we talk about an *equilibrium point*.
- Find regions where the slope is positive;
- Find regions where the slope is negative;

Note: If the equation is *autonomous*, that is, of the form $y' = f(y)$ (not depending explicitly on x), then the direction field constant in the horizontal direction.

Questions to do before the tutorial

1. Find the general solution of the following differential equations.

(a) $\dfrac{dy}{dx} = 1 + \sin x + \sin^2 x$,

Solution: We have

$$y = \int \big(1 + \sin x + \sin^2 x\big) dx = \frac{3}{2}x - \cos x - \frac{1}{4}\sin 2x + C,$$

or, equivalently, $y = \dfrac{3}{2}x - \cos x - \dfrac{1}{2}\sin x \cos x + C$.

(b) $x^3 \dfrac{dy}{dx} = 2x^2 + 5$, $x > 0$,

Solution: $\dfrac{dy}{dx} = 2x^{-1} + 5x^{-3}$, $x > 0$, and so

$$y = \int \left(2x^{-1} + 5x^{-3}\right) dx = 2\ln x - \frac{5}{2x^2} + C.$$

(c) $\dfrac{dy}{dx} = \dfrac{1}{\cosh y}$,

Solution: $\dfrac{dx}{dy} = \cosh y$ and so

$$x = \int \cosh y dy = \sinh y + C.$$

Hence, the general solution is $y = \sinh^{-1}(x - C)$.

Questions to complete during the tutorial

2. Find the particular solutions of the differential equations satisfying the given conditions:

(a) $\dfrac{dy}{dx} = 1 - 2x - 3x^2$, $\quad y(1) = -1$.

Solution: Clearly

$$y = \int (1 - 2x - 3x^2)\, dx = x - x^2 - x^3 + C.$$

With $x = 1$ we get $y = 1 - 1^2 - 1^3 + C = C - 1$, and so the condition implies $C = 0$. Hence the particular solution is

$$y = x - x^2 - x^3.$$

(b) $e^{2x}\dfrac{dy}{dx} + 1 = 0$, $\quad y(x) \to 2$ as $x \to \infty$.

Solution: $\dfrac{dy}{dx} = -e^{-2x}$ and so

$$y = -\int e^{-2x}\, dx = \frac{1}{2}e^{-2x} + C.$$

We see that $y \to C$ as $x \to \infty$, so the condition gives $C = 2$. The particular solution is therefore

$$y = \frac{1}{2}e^{-2x} + 2.$$

(c) $\dfrac{dy}{dx} = \dfrac{y}{2} + \dfrac{1}{2y}, \quad y(0) = 2.$

Solution: We have $\dfrac{dy}{dx} = \dfrac{y^2+1}{2y}$ so $\dfrac{dx}{dy} = \dfrac{2y}{y^2+1}$ and

$$x = \int \frac{2y\,dy}{y^2+1} = \ln(y^2+1) + C.$$

When $y = 2$ we get $x = \ln 5 + C$, and so the condition gives $C = -\ln 5$. Thus the particular solution is

$$x = \ln(y^2+1) - \ln 5 = \ln\Big(\frac{y^2+1}{5}\Big).$$

Solving for y gives $y = \pm\sqrt{5e^x - 1}$. Since the initial condition $y(0) = 2$ is positive the solution of the initial value problem is $y = \sqrt{5e^x - 1}$.

3. According to the Gompertz model, the population N of a colony of animals grows according to the differential equation,

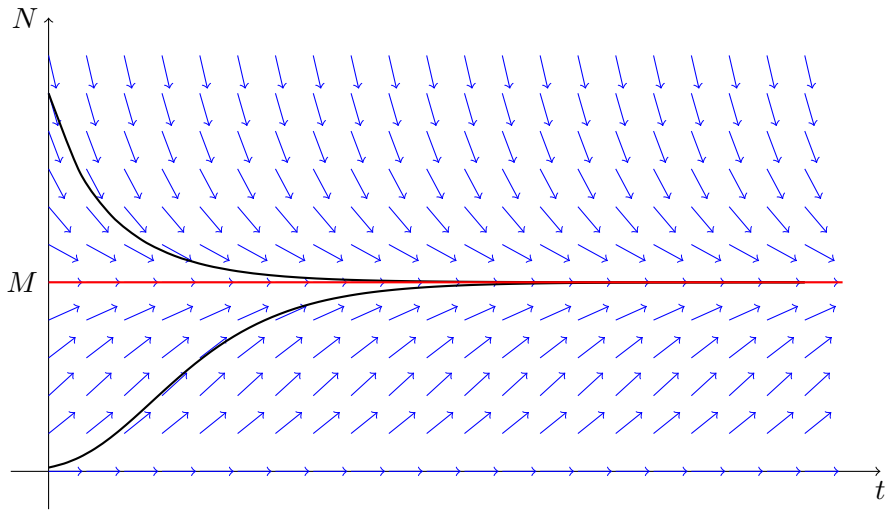
$$\frac{dN}{dt} = -\beta N \ln\Big(\frac{N}{M}\Big),$$

where M is the maximum sustainable population size and β is a positive constant.

(a) Sketch the direction field of the differential equation and some possible solutions.

Solution: If $N = M$, then $N' = 0$, so the direction field is horizontal. If $0 < N < M$, then $N' = -\beta N \ln(N/M) > 0$ as $N/M < 1$. If $N > M$, then $N' = -\beta N \ln(N/M) < 0$ as $N/M > 1$. Also note that $N \ln(N/M) \to 0$ as $N \to 0+$, so the direction field is very flat near the t-axis.

Hence the direction field is as follows. The solution curves are parallel to the direction field at any point.



(b) Set $v := \ln(N/M)$. Show that v satisfies the differential equation $v' = -\beta v$.

Solution: If we differentiate v and use the original differential equation we get

$$v'(t) = \frac{N'(t)}{N(t)} = -\frac{\beta N(t)}{N(t)} \ln\Big(\frac{N(t)}{M}\Big) = -\beta v(t)$$

as claimed.

(c) Solve the differential equation $v' = -\beta v$ and hence find $N(t)$.

Solution: Its solution is $v(t) = Ae^{-\beta t}$ for some constant A as we see by separating variables:

$$\ln|v| = \int \frac{dv}{v} = -\int \beta\,dt = -\beta t + C,$$

so $v = \pm e^{-\beta t + C} = \pm e^C \pm e^{-\beta t} = Ae^{-\beta t}$ for a positive or negative constant. Hence the solution to the original equation is

$$\ln\Big(\frac{N}{M}\Big) = Ae^{-\beta t}$$

It follows that

$$N(t) = M\exp\big(Ae^{-\beta t}\big).$$

(d) Find $\lim_{t\to\infty} N(t)$.

Solution: As $t \to \infty$, we have $Ae^{-\beta t} \to 0$. Therefore $\exp\big(Ae^{-\beta t}\big) \to 1$ and $N(t) \to M$.

(e) Find the particular solution for which $N(0) = M/2$.

Solution: Taking $t = 0$ gives $N = M\exp\big(Ae^0\big) = Me^A$. But we are told that $N(0) = M/2$ and so it follows that $A = -\ln 2$. Thus, the particular solution is

$$N(t) = M\exp\big(-(\ln 2)e^{-\beta t}\big) = M2^{-(e^{-\beta t})}.$$

4. Let y be the number of people in a stable economy who have an income of x or more. The economist Vilfredo Pareto (1848–1923) discovered that the rate at which y decreases with increasing x is directly proportional to the number of people with income x or more and inversely proportional to the income x.

(a) Derive a differential equation for $y(x)$.

Solution: According to Pareto, the rate of change of y with x is directly proportional to y and inversely proportional to x, that is,

$$\frac{dy}{dx} = -k\frac{y}{x}.$$

for some constant k (the proportionality constant). We have put in the negative sign because we are told that y is a decreasing function of x, so $k > 0$.

(b) Find the general solution y in terms of x.

Solution: Separating and integrating the differential equation from the previous gives

$$\int \frac{1}{y}dy = -k\int \frac{dx}{x}$$

and so $\ln y = -k\ln x + C$. Strictly we should have $|y|$ and $|x|$. In the model however we only ever consider $x, y > 0$, so we solve the differential equation for that case. Hence

$$y = e^C x^{-k} = Ax^{-k},$$

where we set $A := e^C$.

(c) Find the particular solution of the differential equation given that the minimum income is x_0 and the total population is N.

Solution: Given that the minimum income is x_0, the whole population must have at least this income, that is, $y(x_0) = N$, so that $N = Ax_0^{-k}$. Substituting A into the general solution gives

$$y = N\Big(\frac{x}{x_0}\Big)^{-k}.$$

5. Find an equation of the curve that passes through $(1, 1)$ and whose slope at (x, y) is y^2/x^3.

Hint: The curve is tangent to the direction field of the differential equation $y' = y^2/x^3$.

Solution: The requirement that the slope of the curve is y^2/x^3 at every point means that

$$\frac{dy}{dx} = \frac{y^2}{x^3},$$

and the fact that $(1, 1)$ is on the curve means that $y(1) = 1$. We solve this initial value problem by separation of variables. Separating and integrating gives

$$\int \frac{dy}{y^2} = \int \frac{dx}{x^3}$$

and hence,

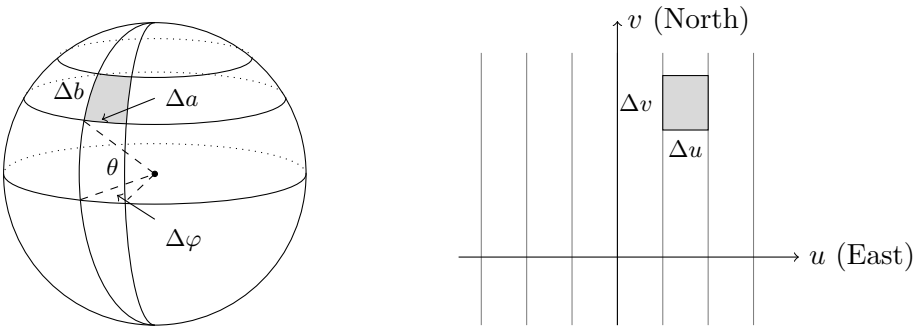
$$-\frac{1}{y} = -\frac{1}{2x^2} + C$$

Hence the general solution is $y(x) = 2x^2/(1 - 2Cx^2)$. We then get $y(1) = 2/(1 - 2C)$, and so the initial condition $y(1) = 1$ implies that $C = -1/2$. So the particular solution describing the required curve is

$$y(x) = \frac{2x^2}{1 + x^2}$$

Extra questions for further practice

6. The Mercator map is one of the most frequently used maps of the earth. It displays the earth such that the parallels and meridians form a rectangular grid. If φ denotes longitude and θ denotes latitude the coordinates of the map are therefore given by $u = u(\varphi)$ and $v = v(\theta)$.



Consider a small rectangle on the sphere of side lengths Δa between longitude φ and $\varphi + \Delta\varphi$, and Δb between latitude θ and $\theta + \Delta\theta$, as shown in the figure. That rectangle is mapped onto a rectangle on the plane with edges parallel to the coordinate axes. The spacing of the parallels on the map is such that the north-south distortion of length is the same as the east-west distortion of length on the map, that is,

$$\frac{\Delta u}{\Delta a} \approx \frac{\Delta v}{\Delta b}.$$

Use this condition to derive a differential equation for $v(\theta)$ and solve it. What initial condition should be assumed?

Solution: Assume the radius of the sphere is R. Then the radius of the parallel at latitude θ is $R\cos\theta$, so $\Delta a = R\Delta\varphi\cos\theta$. The meridians are circles of radius R, so $\Delta b = R\Delta\theta$. By design of the map we choose the spacing of u proportional to that on the equator on the sphere. This means we have $\Delta u = c\Delta\varphi$ for some constant $c > 0$. Hence, applying the condition for the map

$$\frac{c\Delta\varphi}{R\Delta\varphi\cos\theta} = \frac{\Delta u}{\Delta a} \approx \frac{\Delta v}{\Delta b} = \frac{\Delta v}{R\Delta\theta}.$$

Therefore

$$\frac{\Delta v}{\Delta\theta} = \frac{c}{\cos\theta}.$$

If we let $\Delta\theta$ to zero we get the differential equation

$$\frac{dv}{d\theta} = \frac{c}{\cos\theta} = c\sec\theta.$$

Integrating we get

$$v = c\int \sec\theta\, d\theta = c\log(\sec\theta + \tan\theta) + C.$$

We note that

$$\sec\theta + \tan\theta = \frac{1+\sin\theta}{\cos\theta} > 0$$

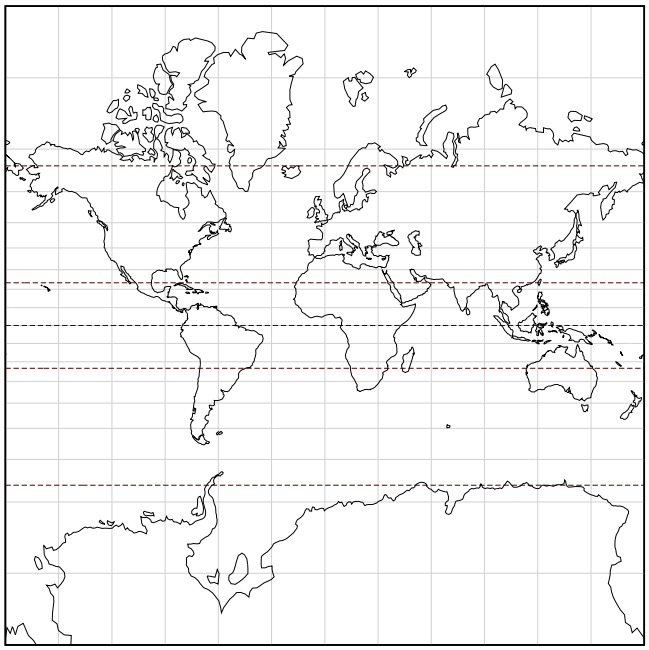
for all $\theta \in (-\pi/2, \pi/2)$ as $\cos\theta > 0$ there. Hence we do not need any absolute value in the logarithm. We assume now that the equator is on the u-axis, so $v(0) = 0$. Hence

$$0 = v(0) = c\log(\sec 0 + \tan 0) + C = c\log 1 + C = C,$$

so $C = 0$. Hence the coordinates for the Mercator map are given by

$$u(\varphi) = c\varphi, \qquad v(\theta) = c\log(\sec\theta + \tan\theta)$$

Applying the formula to the outlines of the continents we get the familiar map below:



7. Consider a particle of mass m in free fall from height h. Let $x(t)$ be its displacement from the initial position and $v(t) = dx/dt$ its velocity at time t.

(a) If we neglect any friction forces, according to Newton's law, v satisfies the differential equation

$$m\frac{dv}{dt} = -mg.$$

(i) Find the solution with initial condition $v(0) = 0$.

Solution: Integrating $m\dfrac{dv}{dt} = -mg$ with respect to time gives $v(t) = -gt + C$ and from $v(0) = 0$ we find $C = 0$. Note that negative velocity means the motion is downward.

(ii) Find the displacement $x(t)$ with initial condition $x(0) = h$.

Solution: Integrating $dx/dt = v = -gt$ with respect to time gives $x(t) = -gt^2/2 + C$ and from $x(0) = h$ we find $C = h$.

(b) Assume now that there is a friction force proportional to the velocity. Then by Newton's law,

$$m\frac{dv}{dt} = -mg - cv$$

for some constant $c > 0$. The negative sign comes from the fact that the force acts in the direction opposite to v.

(i) Find the solution with initial condition $v(0) = 0$.

Solution: This differential equation cannot simply be integrated with respect to time as before. Instead consider the differential equation for the inverse function $t(v)$

$$\frac{dt}{dv} = -\frac{c}{\frac{c}{m}v + g}.$$

Integration gives

$$-t = \frac{m}{c}\log\Big(\frac{c}{m}v + g\Big) + C.$$

Solving for $v(t)$ and renaming $e^{-C} = A$ gives

$$v(t) = -\frac{mg}{c} + A\exp\Big(-\frac{c}{m}t\Big).$$

From the initial condition $v(0) = 0$ we get $0 = -mg/c + A$ so that

$$v(t) = -\frac{mg}{c}\Big(1 - \exp\Big(-\frac{c}{m}t\Big)\Big).$$

Alternatively observe that the equation is of the form $dv/dt + \alpha v = \beta$ with constant α and β. This clearly has the particular solution $v = \beta/\alpha$ Now set $v = u + \beta/\alpha$, then u satisfies $du/dt + \alpha u = 0$, and we know that this has $u = C\exp(-\alpha t)$ as solution. All together this gives $v = C\exp(-\alpha t) + \beta/\alpha$.

(ii) Find the terminal speed $v_\infty = |\lim_{t\to\infty} v(t)|$. Express the constant of proportionality c in terms of v_∞ and write down the solution from the previous part.

Solution: The terminal speed is $v_\infty = |\lim_{t\to\infty} v(t)| = mg/c$. Hence $c = mg/v_\infty$ and $v(t)$ becomes

$$v(t) = -v_\infty\Big(1 - \exp\Big(-\frac{g}{v_\infty}t\Big)\Big).$$

(iii) Find the displacement $x(t)$ with initial condition $x(0) = h$.

Solution: As $dx/dt = v$ we the displacement is

$$x(t) - x(0) = \int_0^t v(s)\,ds$$

and hence

$$x(t) = h - v_\infty\Big[s + \frac{v_\infty}{g}\exp\Big(-\frac{g}{v_\infty}s\Big)\Big]_0^t = h - v_\infty\Big[t + \frac{v_\infty}{g}\exp\Big(-\frac{g}{v_\infty}t\Big) - \frac{v_\infty}{g}\Big].$$

(c) Compute the Taylor polynomials $T_3(t)$ of $x(t)$ for the solutions without and with friction. Verify that for small times they are close to each other.

Solution: The Taylor polynomial $T_3(t)$ for the solution without friction is $x(t) = h - gt^2/2$ (nothing to do), while with friction we find

$$\begin{aligned} x' &= v \\ x'' &= v' = -g - \frac{c}{m}v \\ x''' &= v'' = -\frac{c}{m}v' = -\frac{c}{m}\Big(-g - \frac{c}{m}v\Big) \end{aligned}$$

The computation of the derivatives is simplified by using the differential equation. Now use $x(0) = h$ and $v(0) = 0$ so that

$$x'(0) = 0, \qquad x''(0) = -g, \qquad x'''(0) = \frac{cg}{m}$$

and finally

$$T_3(t) = h - g\frac{t^2}{2} + \frac{cg}{m}\frac{t^3}{3!}$$

Hence the difference is cubic in t. This shows that with friction (and for small t) heavier bodies fall faster.

Alternatively you can use the known series expansion of the exponential function in the solution $x(t)$. The advantage of the above method is that you do not need to know the solution, but you can compute the terms in the Taylor polynomial from the initial conditions and the differential equation. Such a method is of significance in cases where it is not easily possible to get an explicit solution to the given differential equation.

(d) Denote the solution without friction by $v_n(t)$ and the solution with friction by $v_f(t)$. Show that $|v_n(t)| > |v_f(t)|$ for all $t > 0$.

Solution: Since both velocities are negative we need to show that

$$gt > v_\infty(1 - \exp(-gt/v_\infty))$$

for all $t > 0$. Introduce $u = gt/v_\infty$ as a new variable, then the expression becomes

$$v_\infty(u - 1 + \exp(u)) > 0$$

This is obviously true since $e^u > 1$ if $u > 0$.