Material covered

- Homogeneous linear second order differential equations with constant coefficients.
- Inhomogeneous linear second order differential equations with constant coefficients.

Outcomes

After completing this tutorial you should

- be confident in solving homogeneous second order homogeneous and inhomogeneous differential equations in various contexts.

Summary of essential material

Homogeneous linear second order equations with constant coefficients. Consider a differential equation of the form

\[ ay'' + by' + c = 0 \]

with \( a, b, c \in \mathbb{R} \) constants and \( a \neq 0 \). To find the general write down the auxiliary equation

\[ a\lambda^2 + b\lambda + c = 0 \]

and find its roots (real or complex). Depending on the nature of the roots apply the relevant case:

**Case 1:** The auxiliary equation has two distinct real roots \( \lambda_1 \neq \lambda_2 \). Then the general solution is

\[ y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \]

**Case 2:** The auxiliary equation has one (real) double root \( \lambda \). Then the general solution is

\[ y(t) = (A + Bt)e^{\lambda t} \]

**Case 3:** The auxiliary equation has a pair of complex conjugate roots \( \lambda = \mu \pm i\omega \). Then the real form of the general solution is

\[ y(t) = e^{\mu t} (A\cos(\omega t) + B\sin(\omega t)) \]

Inhomogeneous linear second order equations with constant coefficients. Consider a differential equations of the form

\[ ay'' + by' + c = f(t) \]

with \( a, b, c \in \mathbb{R} \) constants and \( a \neq 0 \). The function \( f \) is called the inhomogeneity. The general solution is of the form

\[ y(t) = y_h(t) + y_p(t), \]

where \( y_h \) is the general solution of the homogeneous problem \( ay'' + by' + c = 0 \) and \( y_p \) an arbitrary solution of the inhomogeneous problem we call a particular solution. To find a particular solution we often find a solution that has a similar form to the inhomogeneity \( f \). The idea is to determine the unknown parameters by substituting into the differential equations.

<table>
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<th>Inhomogeneity ( f(t) )</th>
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<tr>
<td>( Ae^{\mu t} )</td>
<td>( Ce^{\mu t} )</td>
<td>( C, D, E, \ldots ) to be determined</td>
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<tr>
<td>( A\cos(\omega t) ) or ( B\cos(\omega t) )</td>
<td>( C\cos(\omega t) + D\sin(\omega t) )</td>
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<td>( At )</td>
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<td>( At^2 )</td>
<td>( Ctf(t) )</td>
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<td>polynomial of degree ( n ) ( f(t) ) solves the homogeneous equation</td>
<td>polynomial of degree ( n )</td>
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Questions to do before the tutorial

1. Find the general solution of each of the following.
   \[ \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} - 5y = 0. \]
   \textbf{Solution:} The auxiliary equation \( \lambda^2 + 4\lambda - 5 = 0 \) has roots \( \lambda = -5, 1 \), and so the general solution is \( y = Ae^{-5x} + Be^x \).

2. Consider the second-order non-homogeneous differential equation \( \frac{d^2 y}{dt^2} + 9y = 0 \).
   \textbf{Solution:} The auxiliary equation \( \lambda^2 + 9 = 0 \) has complex roots \( \lambda = \pm 3i \), and so the general solution is \( y = C\cos 3t + D\sin 3t \).

3. Find the general solution of each of the following.
   \[ \frac{d^2 x}{dt^2} - 6 \frac{dx}{dt} + 9x = 0. \]
   \textbf{Solution:} The auxiliary equation \( \lambda^2 - 6\lambda + 9 = 0 \) has repeated roots \( \lambda = 3, 3 \), and so the general solution is \( x = Ae^{3t} + Bte^{3t} \).

   \[ \frac{d^2 y}{dx^2} - 6 \frac{dy}{dx} + 25y = 0. \]
   \textbf{Solution:} The auxiliary equation \( \lambda^2 - 6\lambda + 25 = 0 \) has complex roots \( \lambda = 3 \pm 4i \), and so the general solution is \( y = e^{3x}(C\cos 4x + D\sin 4x) \).

Questions to complete during the tutorial

4. Solve the following equations, giving the general solution and then the particular solution \( y(x) \) satisfying the given boundary or initial conditions.
   \[ y'' + 4y' + 5y = 0, \quad y(0) = 2, \quad y'(0) = 4 \]
   \textbf{Solution:} The auxiliary equation \( \lambda^2 + 4\lambda + 5 = 0 \) has roots \(-2 \pm i\), and so the general solution is \( y(x) = e^{-2x}(C\cos x + D\sin x) \), which gives \( y'(x) = e^{-2x}((-2C - 2D)\cos x - (C + 2D)\sin x) \). Hence \( y(0) = C \) and \( y'(0) = D - 2C \), so the initial conditions imply \( C = 2 \) and \( D = 8 \), and the particular solution is \( y(x) = 2e^{-2x}(\cos x + 4\sin x) \).
5. We considered the case of a second order differential equation where the auxiliary equation has a double root, say $\lambda_0$. Here we provide an argument why $te^{\lambda_0 t}$ is expected to be a solution. The differential equation in that case is

$$y'' - 2\lambda_0 y' + \lambda_0^2 y = 0.$$ 

The idea is to look at a perturbed equation that has two distinct real roots, then obtain the differential equation in that case is

$$\lambda$$ a double root, say $0$.

(b) Let $h$ and $e^{(\lambda_0 + h)t}$ are solutions to $y'' - (2\lambda_0 + h)y' + \lambda_0(\lambda_0 + h)y = 0$. Briefly explain why

$$e^{(\lambda_0 + h)t} - e^{\lambda_0 t}$$

is a solution of the same perturbed equation.

**Solution:** The auxiliary equation of the given differential equation is

$$0 = \lambda^2 - (2\lambda_0 + h)\lambda + \lambda_0(\lambda_0 + h) = (\lambda - \lambda_0)(\lambda - (\lambda_0 + h)).$$

Hence the roots are $\lambda_0$ and $\lambda_0 + h$ and thus $e^{\lambda_0 t}$ and $e^{(\lambda_0 + h)t}$ are solutions as required. According to the superposition principle, also

$$\frac{1}{h}e^{(\lambda_0 + h)t} - \frac{1}{h}e^{\lambda_0 t} = \frac{e^{(\lambda_0 + h)t} - e^{\lambda_0 t}}{h}$$

is a solution as well.

(b) Let $h \to 0$ in the equation as well as the solution given in part (a) and relate it to the original unperturbed equation. Check that the limit of solutions as $h \to 0$ is a solution to the limit equation.

**Solution:** Applying differentiation with respect to $\lambda$ from first principles we see that

$$\lim_{h \to 0} \frac{e^{(\lambda_0 + h)t} - e^{\lambda_0 t}}{h} = \frac{d}{d\lambda} e^{\lambda t} \bigg|_{\lambda = \lambda_0} = te^{\lambda_0 t}$$

If we let $h \to 0$ in the equation $y'' - (2\lambda_0 + h)y' + \lambda_0(\lambda_0 + h)y = 0$ we obtain the original equation $y'' - 2\lambda_0 y' + \lambda_0 y = 0$. It is not clear that the limit of solutions is a solution of the limit equation, but we might expect this anyway. Hence we need to check by differentiation and substitution. We have, using the chain rule,

$$y(t) = te^{\lambda_0 t}, \quad y'(t) = e^{\lambda_0 t} + \lambda_0 te^{\lambda_0 t}, \quad y''(t) = 2\lambda_0 e^{\lambda_0 t} + \lambda_0^2 te^{\lambda_0 t}.$$

We substitute into the equation to obtain

$$y'' - 2\lambda_0 y' + \lambda_0 y$$

$$= (2\lambda_0 e^{\lambda_0 t} + \lambda_0^2 te^{\lambda_0 t}) - 2\lambda_0(e^{\lambda_0 t} + \lambda_0 te^{\lambda_0 t}) + \lambda_0^2 te^{\lambda_0 t}$$

$$= (2\lambda_0 - 2\lambda_0) e^{\lambda_0 t} + (\lambda_0^2 - 2\lambda_0^2 + \lambda_0^2) te^{\lambda_0 t}$$

$$= 0$$

as expected.
6. First find the general solution of each of the following non-homogeneous second-order differential equations, and then the particular solution for the given initial conditions.

(a) \( y'' + 3y' + 2y = 6e^t \), \( y(0) = 1, y'(0) = 0 \).

**Solution:** The auxiliary equation \( \lambda^2 + 3\lambda + 2 = 0 \) has roots \( \lambda = -1, -2 \), and so the general solution of the homogeneous equation is \( y_h = Ce^{-t} + De^{-2t} \). For a particular solution, try \( y_p = \alpha e^t \). Substituting this into the differential equation gives \( \alpha(e^t + 3e^t + 2e^t) = 6e^t \), which implies \( \alpha = 1 \). So a particular integral is \( y_p = e^t \), and the general solution is

\[
y = Ce^{-t} + De^{-2t} + e^t.
\]

The solution above gives \( y(0) = C + D + 1 \) and \( y'(0) = -C - 2D + 1 \). So \( y(0) = 1 \) and \( y'(0) = 0 \) imply that \( C = -1 \) and \( D = 1 \), and so the required particular solution is \( y = -e^{-t} + e^{-2t} + e^t \).

(b) \( y'' + 3y' + 2y = 6e^{-t} \), \( y(0) = 2, y'(0) = 1 \).

**Solution:** The auxiliary equation and hence the general solution of the homogeneous equation are the same as in the last part. In this case, however, the non-homogeneous term is itself a solution of the homogeneous equation and so we will not be able to produce a particular solution of the form \( \alpha e^{-t} \). The standard procedure in this case is to include a factor \( t \). So a suitable trial solution will take the form \( y_p = \alpha e^{-t}t \). Substitution into the differential equation gives \( \alpha(t-2)e^{-t} + 3\alpha(1-t)e^{-t} + 2\alpha e^{-t} = 6e^{-t} \), which implies \( \alpha = 6 \). So a particular solution is \( y_p = 6te^{-t} \), and the general solution is

\[
y = (6t+C)e^{-t} + De^{-2t}.
\]

The solution above gives \( y(0) = C + D = 6 - C - 2D \). So \( y(0) = 2 \) and \( y'(0) = 1 \) imply that \( C = -1 \) and \( D = 3 \), and so the required particular solution is \( y = (6t-1)e^{-t} + 3e^{-2t} \).

7. (a) For \( \omega \neq 5 \), find the general solution of the non-homogeneous differential equation,

\[
\frac{d^2y}{dt^2} + 25y = 100 \sin \omega t,
\]

and the particular solution subject to the initial conditions \( y(0) = 0 \) and \( \dot{y}(0) = 0 \).

**Solution:** The auxiliary equation \( \lambda^2 + 25 = 0 \) has roots \( \lambda = \pm 5i \), and so the general solution of the homogeneous equation is \( y_h = C \cos 5t + D \sin 5t \). Since the non-homogeneous term is sinusoidal, we try a particular solution of the form, \( y_p = \alpha \sin \omega t + \beta \cos \omega t \). This will work as long as \( \omega \neq \pm 5 \), which we assume for the present. Now, we can save ourselves some trouble by dropping the \( \cos \omega t \) term in \( y_p \). This is permitted because there is no first-order (or any odd-order) derivative term in the differential equation and because only a \( \sin \omega t \) term appears on the right-hand side. (If you have any doubt about this, keep the cosine term in \( y_p \) and find that its coefficient is zero after a calculation.) Substituting \( y_p = \alpha \sin \omega t \) into the differential equation gives \(-\alpha \omega^2 \sin \omega t + 25\alpha \sin \omega t = 100 \sin \omega t \), from which it follows that \( \alpha = 100/(25 - \omega^2) \). Thus, a particular solution is \( y_p = 100(25 - \omega^2)^{-1} \sin \omega t \), and the general solution is

\[
y = C \cos 5t + D \sin 5t + \frac{100}{25 - \omega^2} \sin \omega t.
\]

We want the particular solution such that \( y(0) = \dot{y}(0) = 0 \). Differentiation of the general solution gives

\[
\dot{y} = -5C \sin 5t + 5D \cos 5t + \frac{100\omega}{25 - \omega^2} \cos \omega t.
\]

The initial conditions imply that \( C = 0 \) and \( D = -20\omega/(25 - \omega^2) \). Hence the required particular solution is

\[
y = \frac{100 \sin \omega t - 20 \omega \sin 5t}{25 - \omega^2}.
\]
(b) For \( \omega = 5 \), find a particular solution of the differential equation. Then determine the particular solution with \( y(0) = 0 \) and \( \dot{y}(0) = 0 \).

**Solution:** In the case \( \omega = 5 \), a solution of the form \( y_p = \alpha \sin \omega t + \beta \cos \omega t \) is a solution of the homogeneous equation. The standard trick in this case is to include a factor \( t \), in which case \( y_p = \alpha t \sin 5t + \beta t \cos 5t \). As before, we can simplify the problem by a symmetry argument. Because there is no first-order derivative in the differential equation and because the forcing term is an odd function, we can get away with restricting \( y_p \) to be an odd function. Thus \( y_p = \beta t \cos 5t \). Its derivatives are \( y_p = \beta (-5 \sin 5t + \cos 5t) \) and \( \ddot{y}_p = \beta (-25t \cos 5t - 10 \sin 5t) \). Substituting into the differential equation and cancelling terms shows that \( \beta = -10 \). Hence a particular solution is \( y_p = -10t \cos 5t \), and the general solution is

\[
y = (C - 10t) \cos 5t + D \sin 5t.
\]

Its derivative is \( \dot{y} = (50t - 5C) \sin 5t + (5D - 10) \cos 5t \). The initial conditions are satisfied by \( C = 0 \) and \( D = 2 \). Hence the required particular solution is

\[
y = 2 \sin 5t - 10t \cos 5t.
\]

(c) Find the corresponding particular solution of the differential equation for \( \omega = 5 \) by fixing \( t \) in the result of part (a) and taking the limit as \( \omega \) approaches its special value.

**Solution:** If one puts \( \omega = 5 \) in the result of part (a), the solution becomes a 0/0-type indeterminate form. L’Hôpital’s rule can be used to take the limit \( \omega \to 5 \). Here, we must hold \( t \) constant while we take derivatives with respect to \( \omega \). Thus, in the case of resonance,

\[
y = \lim_{\omega \to 5} \frac{100 \sin \omega t - 20 \omega \sin 5t}{25 - \omega^2} = \lim_{\omega \to 5} \left[ \frac{(\partial/\partial \omega)(100 \sin \omega t - 20 \omega \sin 5t)}{(\partial/\partial \omega)(25 - \omega^2)} \right]_{\omega=5} = \frac{100t \cos 5t - 20 \sin 5t}{-10} = 2 \sin 5t - 10t \cos 5t.
\]

Without L’Hôpital’s rule we can use differentiation from first principles. We can write

\[
y = \lim_{\omega \to 5} \frac{100 \sin \omega t - 20 \omega \sin 5t}{25 - \omega^2} = \lim_{\omega \to 5} \frac{100 \sin \omega t - 100 \sin 5t - 20(\omega - 5) \sin 5t}{25 - \omega^2}
\]

\[
= \frac{100}{5 + \omega} \left[ \frac{\sin \omega t}{\omega - 5} + \frac{20}{5 + \omega} \frac{(\omega - 5) \sin 5t}{\omega - 5} \right]_{\omega=5} = -10t \cos(5t) + 2 \sin 5t
\]

which is the same as before. The factor 10t shows that the amplitude grows without bound.

**Extra questions for further practice**

8. A rope of length \( L \) is suspended at two points \( A \) and \( B \) and hangs freely in-between in such a way that it does not move at all. The rope has constant mass density \( g \) per unit length, that is, a section of length \( \ell \) has mass \( g \ell \). We assume that the rope is perfectly flexible, that is, there is no bending force.

The only forces acting on the rope are the tension force \( T \) tangent to the rope and the gravitational force in the downwards direction. Denote the unit tangent vector along the rope by \( \mathbf{u} \). The height of the rope above ground is given by a function \( y(x) \). Denote acceleration due to gravity by \( g \).
Consider a small section of rope of length $\Delta \ell$ between $x$ and $x + \Delta x$. That section has mass $\rho \Delta \ell$. We denote the unit vectors in the direction of the $x$-axis and the $y$-axis by $i$ and $j$, respectively.

(a) Using the fact that the sum of all forces on $\Delta \ell$ add up to zero, show that

\[
\frac{d}{dx} \left( T(x)u(x) \right) = \rho g \sqrt{1 + \left( y'(x) \right)^2} j.
\]

**Solution:** The length of the section $\Delta \ell$ is given by $\Delta \ell = \sqrt{(\Delta x)^2 + (\Delta y)^2}$, so its mass is $\rho \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Hence the gravitational force on $\Delta \ell$ is given by

\[-\rho \sqrt{(\Delta x)^2 + (\Delta y)^2} j.
\]

The minus sign comes from the fact that the gravitational force points downwards, whereas $j$ points upwards. The other forces on $\Delta \ell$ are the tension forces at the right and left ends. The tension force at the right end is

\[T(x + \Delta x)u(x + \Delta x)\]

and that at the left end is

\[-T(x)u(x)\]

The minus sign comes from the fact that this is a “reaction force” to the section of the rope pulling to the left. The total force on $\Delta \ell$ must be zero, so

\[T(x + \Delta x)u(x + \Delta x) - T(x)u(x) - \rho \sqrt{(\Delta x)^2 + (\Delta y)^2} j = 0.\]

If we rearrange and divide by $\Delta x$ we get

\[\frac{T(x + \Delta x)u(x + \Delta x) - T(x)u(x)}{\Delta x} = \rho \sqrt{1 + \left( \frac{\Delta y}{\Delta x} \right)^2} j.\]

Letting $\Delta x \to 0$, using the definition of a derivative, we get the required differential equation.

(b) Explain why the unit tangent vector $u$ is given by

\[u(x) = \frac{1}{\sqrt{1 + (y'(x))^2}} i + \frac{y'(x)}{\sqrt{1 + (y'(x))^2}} j.\]
Solution: The slope of the tangent at every point is given by \( y'(x) \). Hence the vector \( \mathbf{i} + y'(x)\mathbf{j} \) points in the direction of \( \mathbf{u}(x) \). To get the unit vector we need to divide by the length which is given by \( \sqrt{1 + (y'(x))^2} \). Hence
\[
\mathbf{u}(x) = \frac{1}{\sqrt{1 + (y'(x))^2}} \mathbf{i} + \frac{y'(x)}{\sqrt{1 + (y'(x))^2}} \mathbf{j}.
\]

(c) By considering the component of the differential equation from (a) in the \( x \)-direction, that is, the direction of \( \mathbf{i} \), show that
\[
T(x) = H \sqrt{1 + (y'(x))^2}
\]
for some constant \( H \). Give a physical interpretation of \( H \).

Solution: According to part (b) the horizontal component of \( \mathbf{u} \) is given by
\[
\frac{1}{\sqrt{1 + (y'(x))^2}}
\]
Hence the horizontal component of (a) is given by
\[
\frac{d}{dx} \left( \frac{T(x)}{\sqrt{1 + (y'(x))^2}} \right) = 0.
\]
Hence, there exists a constant \( H \) so that
\[
\frac{T(x)}{\sqrt{1 + (y'(x))^2}} = H,
\]
and therefore \( T(x) = H \sqrt{1 + (y'(x))^2} \) as claimed.

The horizontal component of the tension force is
\[
\frac{T(x)}{\sqrt{1 + (y'(x))^2}}
\]
Using the explicit expression of \( T \) the horizontal component of the tension force has the constant value \( H \).

(d) By considering the component of the differential equation from (a) in the \( y \)-direction, that is, the direction of \( \mathbf{j} \), show that
\[
y''(x) = \frac{\rho g}{H} \sqrt{1 + (y'(x))^2}.
\]

Solution: According to part (b) the vertical component of \( \mathbf{u} \) is given by
\[
\frac{y'(x)}{\sqrt{1 + (y'(x))^2}}
\]
Hence the vertical component of (a) is given by
\[
\frac{d}{dx} \left( \frac{T(x)y'(x)}{\sqrt{1 + (y'(x))^2}} \right) = \rho g \sqrt{1 + (y'(x))^2}.
\]
Substituting the solution from (c) we get
\[
\frac{d}{dx} (Hy'(x)) = Hy''(x) = \rho g \sqrt{1 + (y'(x))^2}.
\]
If we divide by \( H \) we get the required differential equation.
(e) Find the general solution of the differential equation in (d). Note that the differential equation is a first order differential equation for \( z(x) = y'(x) \).

**Solution:** Rewriting the original differential equation as a differential equation for \( z(x) = y'(x) \) we get

\[
\frac{dz}{dx} = \frac{\varrho g}{H} \sqrt{1 + z^2}.
\]

We first separate variables and write

\[
\frac{dz}{\sqrt{1 + z^2}} = \frac{\varrho g}{H} \, dx
\]

and integrating we get

\[
\int \frac{dz}{\sqrt{1 + z^2}} = \int \frac{\varrho g}{H} \, dx = \frac{\varrho gx}{H} + C.
\]

For the integral on the left hand side we use the substitution \( z = \sinh t \). Then \( \frac{dz}{\sqrt{1 + z^2}} = \cosh t \, dt \). Using that \( 1 + \sinh^2 t = \cosh^2 t \) we get

\[
\frac{\varrho gx}{H} + C = \int \frac{\cosh t}{\sqrt{1 + \sinh^2 t}} \, dt = \int \frac{\cosh t}{\cosh t} \, dt = t = \sinh^{-1} z.
\]

We do not need a constant as that constant can be merged with \( C \). Alternatively we could use a table of standard integrals to evaluate the integral. Hence

\[
z = \sinh \left( \frac{\varrho gx}{H} + C \right).
\]

Next we recall that \( z = y' \), so

\[
y(x) = \int z(x) \, dx = \int \sinh \left( \frac{\varrho gx}{H} + C \right) \, dx = \frac{H}{\varrho g} \cosh \left( \frac{\varrho gx}{H} + C \right) + D.
\]

The cosh curve is often called the catenary. The constants \( C, D \) and \( H \) could be computed in terms of the length \( L \) the mass density \( \varrho \) and the coordinates of \( A \) and \( B \), but this is rather tedious to do for the general situation.

9. Find the general solution of the differential equation

\[
y'' - 2y' + 5y = 0,
\]

expressing your answer in real form. What is the particular solution satisfying \( y(0) = 1 \) and \( y(\pi/4) = 2 \)?

**Solution:** The auxiliary equation is \( \lambda^2 - 2\lambda + 5 = 0 \), which has roots \( \lambda = 1 \pm 2i \), and so the general solution is

\[
y = e^t (A \cos 2t + B \sin 2t).
\]

Hence \( y(0) = E \) and \( y(\pi/4) = e^{\pi/4}F \). If \( y(0) = 1 \) and \( y(\pi/4) = 2 \) then \( A = 1 \) and \( B = 2e^{-\pi/4} \), and hence the particular solution is

\[
y = e^t \left( \cos 2t + 2e^{-\pi/4} \sin 2t \right).
\]

10. Solve the following equations, giving the general solution and then the particular solution \( y(x) \) satisfying the given boundary or initial conditions.

(a) \( 2y'' - 7y' + 5y = 0, \quad y(0) = 1, \ y'(0) = 1 \)

**Solution:** The auxiliary equation \( 2\lambda^2 - 7\lambda + 5 = 0 \) has roots \( 5/2 \) and \( 1 \), and so the general solution is \( y(x) = Ae^{5x/2} + Be^x \), which gives \( y'(x) = (5A/2)e^{5x/2} + Be^x \). Hence \( y(0) = A + B \) and \( y'(0) = (5A/2) + B \), so the initial conditions imply \( A = 0 \) and \( B = 1 \), and the particular solution is \( y(x) = e^x \).
(b) \( y'' + 4y' + 3y = 0, \quad y(-2) = 1, \ y(2) = 1 \)

**Solution:** The auxiliary equation \( \lambda^2 + 4\lambda + 3 = 0 \) has roots \(-1\) and \(-3\), and so the general solution is \( y(x) = Ae^{-x} + Be^{-3x} \). Hence \( y(-2) = Ae^{-2} + Be^{6} \) and \( y(2) = Ae^{-2} + Be^{-6} \), so the boundary conditions imply \( Ae^{2} + Be^{6} = 1 \) and \( Ae^{-2} + Be^{-6} = 1 \). Solving these simultaneous equations gives

\[
A = \frac{\sinh 6}{\sinh 4} = 7.3915, \quad B = \frac{2}{\sinh 4} = -0.1329,
\]

and so the particular solution satisfying the boundary conditions is

\[
y(x) = 7.3915e^{-x} - 0.1329e^{-3x}.
\]

(c) \( 2y'' - 2y' + 5y = 0, \quad y(0) = 0, \ y(2) = 2 \)

**Solution:** The auxiliary equation \( 2\lambda^2 - 2\lambda + 5 = 0 \) has roots \((1 \pm 3i)/2\), and so the general solution is \( y(x) = e^{x/2}(A\cos(3x/2) + B\sin(3x/2)) \). Hence \( y(0) = A \), and the first boundary condition implies \( A = 0 \). Thus \( y(2) = B\sin 3 \), and so the second boundary condition implies \( B = 2/(\sin 3) = 5.2137 \), and hence the particular solution satisfying the boundary conditions is \( y(x) = 5.2137e^{x/2}\sin(3x/2) \).

(d) \( y'' - 4y' + 4y = 0, \quad y(0) = -2, \ y(1) = 0 \)

**Solution:** The auxiliary equation \( \lambda^2 - 4\lambda + 4 = 0 \) has only one double root \( \lambda = 2 \), and so the general solution is \( y(x) = (A + Bx)e^{2x} \). Hence \( y(0) = A \) and the first boundary condition implies \( A = -2 \). Thus \( y(1) = (-2 + B)e^{2} \), and so the second boundary condition implies \( B = 2 \), and hence the particular solution satisfying the boundary conditions is \( y(x) = 2(x - 1)e^{2x} \).

11. Find the particular solution of the differential equation \( y'' - 6y' + 9y = e^{3x} \) which satisfies the initial conditions \( y(0) = 1 \) and \( y'(0) = 0 \).

**Solution:** The auxiliary equation of the homogeneous problem is \( \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0 \). As \( \lambda = 3 \) is a double root \( e^{3x} \) and \( xe^{3x} \) solve the homogeneous equation. Hence the inhomogeneity \( e^{3x} \) solves the equation. Normally we would find a particular solution of the form \( Ax e^{3x} \), but that is a solution of the homogeneous equation as well. Hence we multiply by another \( x \) and try a particular solution of the form \( y = Ax^2 e^{3x} \). We note that \( y'(x) = 2x Ax e^{3x} + 3x^2 Ae^{3x} \) and \( y''(x) = 2A e^{3x} + 12Ax e^{3x} + 9A x^2 e^{3x} \). Substitution into the equation yields

\[
2A e^{3x} + 12Ax e^{3x} + 9x^2 Ae^{3x} - 6(2Ax e^{3x} + 3A x^2 e^{3x}) + 9Ax^2 e^{3x} = e^{3x}.
\]

If we cancel \( e^{3x} \neq 0 \) and collect terms according to powers of \( x \) we obtain

\[
2A + (12A - 12A)x + (9A - 18A + 9A)x^2 = 2A = 1
\]

Hence \( A = \frac{1}{2} \) and the general solution is

\[
y = \left( C + Dx + \frac{x^2}{2} \right) e^{3x}.
\]

To make use of the initial conditions note that

\[
y' = \left( 3C + 3Dx + \frac{3x^2}{2} + D + x \right) e^{3x}.
\]

Hence \( y(0) = C \) and \( y'(0) = 3C + D \). So the conditions \( y(0) = 1 \) and \( y'(0) = 0 \) imply that \( C = 1 \) and \( D = -3 \). Hence, the required particular solution is

\[
y = \left( 1 - 3x + \frac{x^2}{2} \right) e^{3x}.
\]