The University of Sydney
School of Mathematics and Statistics

Tutorial for Week 13

MATH1903: Integral Calculus and Modelling (Advanced) — Semester 2, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1903/
Lecturers: Daniel Daners and David Easdown

Material covered

☐ systems of linear differential equations
☐ matrix exponentials (optional)

Outcomes

After completing this tutorial you should

☐ be able to solve systems of two linear differential equations either by reducing them to a second order equation, or by using the eigenvalues and eigenvectors of the system matrix.

Summary of essential material

Linear system of differential equations can be solved in the same way as systems of algebraic equations: eliminate one variable and solve for the other, then substitute back and solve for the one eliminated. The way to eliminate one of the unknown function is achieved by differentiating one of the equations. As an example consider the system

$$x' = 7x - 5y \qquad y' = 4x - 5y$$

to a second order equation for either x or y, and then find the general solution of the system.

Differentiate the first equation and then substitute the second equation to get

$$x'' = 7x' - 5y' = 7x' - 5(4x - 5y)$$

From the first equation we have $5y = 7x - x'$, so

$$x'' = 7x' - 5(4x - 5y)7x' - 5(4x - 7x + x') = 2x' + 15x.$$

Hence the equation for x is $x'' - 2x' - 15x = 0$. Its auxiliary equation is $\lambda^2 - 2\lambda - 15 = (\lambda - 5)(\lambda + 3) = 0$. The general solution of the second order equation for x therefore is

$$x(t) = Ae^{5t} + Be^{-3t}$$

From the first differential equation we can determine $y(t)$, namely

$$5y = 7x - x' = 7Ae^{5t} + 7Be^{-3t} - 5Ae^{5t} + 3Be^{-3t} = 2Ae^{5t} + 10e^{-3t}$$

Hence the solution of the system is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A \begin{bmatrix} 1 \\ \frac{2}{5} \end{bmatrix} e^{5t} + B \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-3t}.$$

Replacing the constant A by a different constant (again denoted by A) we could write the general solution as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{5t} + B \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-3t}.$$

There is a problem if we first eliminate y to get x, and then eliminate y to get x. The general solutions in each case involves two constants, so there would be four constants. Given we have two equations with two unknowns we only need two. Hence these constants depend on each other. Therefore only do one elimination (either x or y), then get the other by back substitution as shown in the example above.

Questions to complete during the tutorial

1. Find the general solution of the system of differential equations

$$\frac{dx}{dt} = 5x - 3y, \qquad \frac{dy}{dt} = 2x$$

by differentating the first equation and then using the second equation to form a linear second order differential equation for $x(t)$. Then find the particular solutions satisfying the initial conditions $x(0) = 2$ and $y(0) = 1$.

2. Find the general solution of the system

$$\dot{x} = 2x - y, \qquad \dot{y} = x + 2y$$

by reducing the system to a second order equation.

3. Consider the pair of differential equations

$$\frac{dx}{dt} = 7x - 2y, \qquad \frac{dy}{dt} = 2x + 3y.$$

(a) Obtain a second-order differential equation for $x(t)$ and find its general solution.

(b) Find the associated general solution for $y(t)$.

(c) Show that, if $x(0) > y(0)$, then $x(t)$ and $y(t)$ increase without limit as $t \to \infty$. Conversely, if $x(0) < y(0)$, show that $x(t)$ and $y(t)$ decrease without limit as $t \to \infty$.

(d) Find the particular solution for $x(t)$ and $y(t)$ satisfying the initial conditions $x(0) = 2$, $\dot{y}(0) = 1$.

(e) Confirm explicitly that the expressions you have obtained for $x(t)$ and $y(t)$ obey the first order equations given in part (a).

4. Let $u(t)$ and $v(t)$ model the sizes of a population of predator and prey, respectively (such as foxes and rabbits). The size of the populations are governed by a system of differential equation of the form

$$\frac{du}{dt} = (-28 + 7v)u, \qquad \frac{dv}{dt} = (-4u + 20)v.$$

(a) Explain the significance of the terms in this system of equations.

(b) Sketch the direction field for the system of equations. First look at where the field is horizontal or vertical, then fill in the rest.

Extra questions for further practice

5. Find the general solution of the pair of differential equations,

$$\frac{dx}{dt} = 5x - 3y, \qquad \frac{dy}{dt} = 2x,$$

by differentiating the first equation and then using the second to obtain a linear second-order differential equation for $x(t)$. (Your solution should have *two* arbitrary constants of integration.) Find the particular solution satisfying the initial conditions $x = 2$, $y = 1$ when $t = 0$.

6. The following is a model for an arms race between two superpowers X and Y. Denote the level of preparation of X for war by $x(t)$ and that of Y by $y(t)$, where t represents time. The model consists of the equations,

$$\frac{dx}{dt} = -ax + by, \qquad \frac{dy}{dt} = cx - dy,$$

where a, b, c and d are positive constants.

(a) Eliminate y to obtain a second-order homogeneous linear differential equation for $x(t)$.

(b) Write down the auxiliary equation of this differential equation, and its general solution in terms of its roots λ_1 and λ_2. Show that if $ad > bc$ then both roots of the auxiliary equation must be negative. What does this suggest about the likelihood of war?

(c) Suppose $a = d = 1$, $b = c = 3$, and that $x = 5$, $y = 1$ at the initial time $t = 0$. Find the particular solution for x and y. What can you conclude about the likelihood of war in this case?

(d) Briefly discuss the assumptions underlying the model. Do you think these assumptions are realistic?

7. An electrical circuit comprises a capacitor, a resistor, and an inductor connected in series to a voltage generator with a prescribed voltage $V(t)$. In terms of the electric charge $Q(t)$ and the current $I(t) = dQ/dt$, the voltage drops across the components are Q/C for a capacitance C, IR for a resistance R, and LdI/dt for an inductance L. The total voltage drop is equal to that supplied by the generator,

$$\frac{Q}{C} + IR + L\frac{dI}{dt} = V(t).$$

(a) Using $I = dQ/dt$, derive the equation,

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \frac{dV}{dt}.$$

(b) Find the general solution of the homogeneous equation, distinguishing three possible types of behaviour, depending on the values of $R^2 - 4L/C$. Show that, provided $L, R > 0$, all solutions die out as $t \to \infty$.

(c) Find a particular solution for an AC voltage source $V(t) = V_0 \cos(\omega_0 t)$.

Challenge questions (optional)

In lectures we showed that solutions to a linear systems of differential equations can be obtained by using methods of linear algebra, in particular eigenvalues and eigenvectors. In vector notation we saw that a system takes the form

$$\frac{d}{dt}\boldsymbol{u}(t) = M\boldsymbol{u}(t)$$

with M being a matrix. Such an equation looks like the standard equation $x' = mx$ in one dimension. Its solution is

$$x(t) = Ce^{mt}.$$

We can try a similar idea for systems and write a matrix exponential

$$\boldsymbol{u}(t) = e^{Mt}.$$

But what is a matrix exponential? We can define it by means of the *exponential series*

$$e^{Mt} = \sum_{k=0}^{\infty} \frac{t^k}{k!} M^k.$$

It turns out that this series converges. In the following exercises you find examples how to compute theses exponentials using your knowledge from first semester linear algebra MATH1902.

8. The system of differential equations from Question 1 can alternatively be written in vector form

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

If we set $u(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and denote the matrix by M the above can be written as $\boldsymbol{u}' = M\boldsymbol{u}$. The solution with initial condition $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \boldsymbol{u}_0$ can be written as the matrix exponential $\boldsymbol{u}(t) = e^{Mt}\boldsymbol{u}_0$. In practice, computing the matrix exponential is often unnecessary, and the general solution is sufficent. If desired the matrix exponential can be obtained from that.

(a) Compute the eigenvalues and eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ of the matrix M by first finding the roots λ_1, λ_2 of the characteristic polynomial $\det(M - \lambda I)$.

(b) From the previous part $Ae^{\lambda_1 t}\boldsymbol{v}_1 + Be^{\lambda_2 t}\boldsymbol{v}_2$ is the general solution of the system of equations. Determine the solution with initial values $x(0) = 2$ and $y(0) = 1$. Compare the result with that in Question 1.

(c) Here we want to describe a way to get the matrix exponential e^{tM}. Form the matrix $X(t) := \begin{bmatrix} e^{\lambda_1 t}\boldsymbol{v}_1 & e^{\lambda_2 t}\boldsymbol{v}_2 \end{bmatrix}$ with columns given by $e^{\lambda_2 t}\boldsymbol{v}_2$ and $e^{\lambda_t 2}\boldsymbol{v}_2$.

(i) Show that $\frac{d}{dt}X(t) = MX(t)$, where differentiation of matrices is done on each entry.

(ii) Assuming that the solution of $\boldsymbol{u}' = M\boldsymbol{u}$ with given initial condition $\boldsymbol{u}(0) = \boldsymbol{u}_0$ is unique, show that $e^{Mt} = X(t)X_0^{-1}$. (For the uniqueness, see Question 10.)

(d) Use the above method to compute e^{Mt}.

9. Consider the system of differential equations $x' = -x - y$, $y' = x - 3y$. Denote by $M = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$ the matrix associated with that system.

(a) Show that M has a double eigenvalue $\lambda = -2$ and one independent eigenvector $\boldsymbol{v}$ only.

(b) To find a second solution try a solution of the form $(\boldsymbol{w} + t\boldsymbol{v})e^{-2t}$, where $\boldsymbol{w}$ is to be determined and $\boldsymbol{v}$ is the eigenvector.

(c) Use the idea from Question 1(c) to compute e^{Mt}.

10. We show the uniqueness of the solution of $\boldsymbol{u}' = M\boldsymbol{u}$, where M is a matrix.

(a) From the definition of the matrix exponential, show that $Me^{tM} = e^{tM}M$. Differentiate $e^{-tM}e^{tM}$ and hence show that $e^{-tM}e^{tM} = I$ is constant.

(b) Suppose that $X(t)$ is a solution. Differentiate $e^{-tM}X(t)$ and hence show it is constant. Deduce that the solution is unique and that $e^{tM} = X(t)X(0)^{-1}$