THE UNIVERSITY OF SYDNEY

MATH1903 INTEGRAL CALCULUS AND MODELLING (ADVANCED)

Semester 2 | **Tutorial Solutions Week 10** | 2009

1. Which of the following differential equations are linear? Can any of the nonlinear cases be transformed into a linear differential equation by a simple change of variables? (You are not required to solve these DEs; however, all have elementary solutions and you should attempt them at home after the tutorial.)

(a) $\dfrac{dy}{dx} + \dfrac{3y}{x} = \sin x$

(b) $(x-1)^3\dfrac{dy}{dx} + 4(x-1)^2 y = x+1$

(c) $(x-y)\dfrac{dy}{dx} + y = e^x$

(d) $\dfrac{dy}{dx} = \dfrac{y^2+1}{2xy+1}$

Solution

A linear equation of the first-order is one that can be written in the form, $\dfrac{dy}{dx} + P(x)y = Q(x)$.

(a) $\dfrac{dy}{dx} + \dfrac{3}{x}y = \sin x$ is linear.

(b) $\dfrac{dy}{dx} + \dfrac{4}{x-1}y = \dfrac{x+1}{(x-1)^3}$ is linear.

(c) $\dfrac{dy}{dx} = \dfrac{y-e^x}{y-x}$ is not linear. However, the change of variable $y = \bar{y} + x$ reduces this DE to a separable equation. A further change of variable $w = \bar{y}^2$ yields the linear DE, $w' = 2(x - e^x)$.

(d) $\dfrac{dy}{dx} = \dfrac{y^2+1}{2xy+1}$ is not linear. However, the reciprocal equation, $\dfrac{dx}{dy} - \dfrac{2y}{y^2+1}x = \dfrac{1}{y^2+1}$ **is** linear, and can be solved as such.

The question did not ask for solutions. Nevertheless, all these equations have elementary solutions and here they are:

(a) $y = \dfrac{6-x^2}{x^2}\cos x + \dfrac{3(x^2-2)}{x^3}\sin x + \dfrac{C}{x^3}$

(b) $y = \dfrac{x^3 - 3x + C}{3(x-1)^4}$

(c) $y = x \pm \sqrt{x^2 - 2e^x + C}$

(d) $2x = y + (y^2+1)(\tan^{-1} y + C)$

2. Solve

(a) $\dfrac{dy}{dx} - y\tan x = x$

(b) $\dfrac{dx}{dt} + 2tx = 2t^3$

(c) $dx - (\sec y + 2x\tan y)\,dy = 0$

(d) $\dfrac{dy}{dx} = \dfrac{2y}{y - x - y^3}$

(e) $(1+x)\dfrac{dy}{dx} + y = 3x^2$, given $y(0) = 2$.

(f) $2\,dx + (2x+3y)\,dy = 0$, given $y(2) = 0$.

Solution

(a) Linear in y. The integrating factor (I.F.) is $\cos x$, so $d\big((\cos x)y\big)/dx = x\cos x$. Integration by parts gives $(\cos x)y = x\sin x + \cos x + C$ and hence $y = x\tan x + 1 + C\sec x$.

(b) Linear in x. The integrating factor is e^{t^2}, so $d(e^{t^2}x)/dt = 2t^3e^{t^2}$. Integration by parts gives $e^{t^2}x = (t^2-1)e^{t^2} + C$ and hence $x = t^2 - 1 + Ce^{-t^2}$.

(c) Linear in x: $dx/dy - (2\tan y)x = \sec y$. The integrating factor is $\cos^2 y$, so $d(x\cos^2 y)/dy = \cos y$, leading to $x = (\sin y + C)/\cos^2 y$. Optional inverse: $y = \sin^{-1}\big\{(-1 \pm \sqrt{4x^2 - 4Cx + 1}\,)/(2x)\big\}$.

(d) Linear in x: $dx/dy + x/(2y) = (1-y^2)/2$. The integrating factor is $y^{1/2}$, leading to the general solution, $x = (1/3)y - (1/7)y^3 + Cy^{-1/2}$.

(e) Linear in y: $dy/dx + (1+x)^{-1}y = 3x^2(1+x)^{-1}$. The integrating factor is $1+x$. So $d\big((1+x)y\big)/dx = 3x^2$, leading to $y = (x^3+C)/(1+x)$. Imposing the condition $y = 2$ when $x = 0$ gives $C = 2$ and so the particular solution is $y = (x^3+2)/(1+x)$.

(f) Linear in x: $dx/dy + x = -3y/2$. The integrating factor is e^y, so $d(xe^y)/dy = -3ye^y/2$, leading to $2x = -3y + 3 + Ce^{-y}$. Imposing the condition $y = 0$ when $x = 2$ gives $C = 1$, and so the particular solution is $e^{-y} = 2x + 3y - 3$.

3. A tank initially contains 700 litres of fresh water. A pipe is opened which admits salty water at 10 litres/min. At the same time, a drain is opened to allow 8 litres/min of the mixture to leave the tank. If the inflowing salty water contains 0.01 kg of salt per litre, what is the mass of salt in the tank after 60 minutes? What is the concentration of the salt?

Solution

Let $m(t)$ be the mass of salt (in kilograms) in the tank at time t. Since the tank initially contains fresh water, $m(0) = 0$. In this problem, the volume $V(t)$ of the water in the tank is not constant since water is leaving the tank at a slower rate than the water entering. The net gain is 2 litres/min. Hence the volume of the water in the tank at time t is $V(t) = 700 + 2t$. The concentration of salt in the water leaving the tank is $m(t)/V(t)$. The differential equation governing the mass $m(t)$ is given by

$$\begin{aligned}\frac{dm}{dt} &= \{\text{rate in}\} - \{\text{rate out}\} \\ &= 10 \times 0.01 - 8 \times m/V \\ &= \frac{1}{10} - \frac{8m}{700+2t}, \\ \frac{dm}{dt} + \frac{8m}{700+2t} &= \frac{1}{10}.\end{aligned}$$

This is a linear differential equation of the first order. Its integrating factor is

$$I(t) = e^{\int 8/(700+2t)\,dt} = e^{4\ln(700+2t)} = (700+2t)^4.$$

Hence, the equation can be written in the exactly integrable form,

$$\frac{d}{dt}\{(700+2t)^4 m\} = \frac{1}{10}(700+2t)^4.$$

Integrating gives

$$(700+2t)^4 m = \frac{(700+2t)^5}{10\cdot 2\cdot 5} + C = \frac{(700+2t)^5}{100} + C.$$

The initial condition $m(0) = 0$ implies that $C = -(700)^5/100 = -7\cdot(700)^4$. Hence, the required particular solution of the differential equation is

$$m(t) = \frac{700+2t}{100} - \frac{7\cdot(700)^4}{(700+2t)^4}.$$

After 60 minutes, the mass of salt is

$$m(60) = \frac{820}{100} - \frac{7\cdot(700)^4}{(820)^4} = 8.2 - 3.717 = 4.483 \text{ kg}.$$

The corresponding concentration is $m(60)/V(60) = 4.483/820 = 0.00547$ kg/L.

4. Some rocks contain a radioactive isotope of radium, Ra^{226}, which has a half-life of 1590 years and decays into an isotope of lead, Pb^{210}. This lead isotope is itself radioactive, and decays with a half-life of 22 years. Let $R(t)$ be the amount of radium in the rock and $L(t)$ be the amount of lead. Then the rate of change of L is the rate at which lead is produced by the decay of radium, minus the rate at which the lead decays; so $dL/dt = \lambda R - \mu L$ where λ and μ are the decay constants of radium and lead respectively. Given that $R = R_0 e^{-\lambda t}$ and that $L = 0$ at $t = 0$, solve this equation to show that

$$L(t) = \frac{\lambda R_0}{\mu - \lambda}(e^{-\lambda t} - e^{-\mu t}).$$

What are the values of λ and μ?

Solution

The differential equation,

$$\frac{dL}{dt} = \lambda R_0 e^{-\lambda t} - \mu L,$$

is linear, with $P(t) = \mu$ so that the integrating factor is $I(t) = e^{\int \mu\, dt} = e^{\mu t}$. Multiplying through by this factor, we can put the DE in the form,

$$\frac{d}{dt}\left(e^{\mu t} L\right) = \lambda R_0 e^{\mu t} e^{-\lambda t} = \lambda R_0 e^{(\mu-\lambda)t}.$$

Integrating gives

$$e^{\mu t} L = \frac{\lambda R_0}{\mu - \lambda} e^{(\mu-\lambda)t} + C, \quad \text{or} \quad L = \frac{\lambda R_0}{\mu - \lambda} e^{-\lambda t} + C e^{-\mu t}.$$

Putting $L = 0$ when $t = 0$, we find that $C = -\lambda R_0/(\mu - \lambda)$ and so

$$L(t) = \frac{\lambda R_0}{\mu - \lambda}\left(e^{-\lambda t} - e^{-\mu t}\right).$$

In each case, the decay constant is given by $k = (\ln 2)/T_{1/2}$, where $T_{1/2}$ is the half-life. Thus $\lambda = 0.6931/1590 = 4.36 \times 10^{-4}$, and $\mu = 0.6931/22 = 3.15 \times 10^{-2}$.

5. (a) Obtain first-order differential equations that govern the following one-parameter families of curves:

(i) $\quad y = Cx^4;$ $\qquad\qquad$ (ii) $\quad \dfrac{x^2}{C} + \dfrac{y^2}{C-1} = 1.$

(b) Find the families of curves that are orthogonal to the families in part (a). Describe the family of curves obtained in the first case. In the second case, the given curves are ellipses when $C > 1$ and hyperbolas when $0 < C < 1$. Show that the ellipses are orthogonal to the hyperbolas. If (x_0, y_0) is any point in the interior of the first quadrant, find the two values of C giving the ellipse and hyperbola through this point.

Solution

(a) (i) For the class of curves $y = Cx^4$ (quartic parabolas), eliminate the parameter C by differentiation:

$$C = \frac{y}{x^4} \quad \text{implies} \quad \frac{d}{dx}\left(\frac{y}{x^4}\right) = 0.$$

Taking the derivative and clearing the denominator, we arrive at the differential equation,

$$x\frac{dy}{dx} - 4y = 0.$$

(*ii*) For the class of confocal ellipses and hyperbolas $x^2/C + y^2/(C-1) = 1$, rearrange as a quadratic equation for the parameter:

$$C^2 - (x^2 + y^2 + 1)C + x^2 \;=\; 0.$$

To eliminate C, one could solve for C and then differentiate, but that method would be slow. A better way is to differentiate the quadratic equation directly with respect to x, thereby eliminating C^2, and then solve for C:

$$C \;=\; \frac{x}{x + yy'}\,, \qquad C - 1 \;=\; -\,\frac{yy'}{x + yy'}\,.$$

Substitute these into the original equation to get the differential equation:

$$\big(xy' - y\big)\big(x + yy'\big) \;=\; y'.$$

(b) (*i*) Given a one-parameter family of smooth curves that is governed by the differential equation $F(x, y, y') = 0$, the orthogonal (i.e., perpendicular) family of curves is governed by the differential equation $F(x, y, -1/y') = 0$, i.e., one replaces dy/dx by $-1/(dy/dx)$ in the differential equation. In the case of the family of quartic parabolas $y = Cx^4$, the differential equation was found above to be $xy' - 4y = 0$. Hence, the orthogonal family of curves satisfies

$$x + 4yy' \;=\; 0.$$

This is a separable differential equation (and the variables are already separated). Integrating gives

$$x^2 + 4y^2 \;=\; K^2,$$

where K is the constant of integration. This is a family of concentric ellipses having semiaxes K and $K/2$.

(*ii*) In the case of the family of ellipses and hyperbolas, the governing differential equation was found above to be

$$\big(xy' - y\big)\big(x + yy'\big) \;=\; y'.$$

When one replaces y' by $-1/y'$, this equation is found to be unchanged. This means that, in some sense, the family of ellipses and hyperbolas is orthogonal to itself. More specifically, we will show that the family of ellipses $C > 1$ is orthogonal to the family of hyperbolas $0 < C < 1$. (We are excluding the degenerate cases $C = 0$ and $C = 1$, which are straight lines or parts thereof along the coordinate axes.) Although, it is not necessary for the solution of this problem, it is worth noting that these ellipses and hyperbolas are *confocal*, i.e., they all have the same two foci. If $a > b$, the ellipse $x^2/a^2 + y^2/b^2 = 1$ has foci on the x-axis (the major axis) at $(\pm\sqrt{a^2 - b^2}\,, 0)$. In the case $a^2 = C$ and $b^2 = C - 1$, the foci are at $(\pm 1, 0)$, and so are the same for all C. Similarly, the foci of the hyperbolas are also at $(\pm 1, 0)$.

Choose any point (x_0, y_0) not on the coordinate axes. Without loss of generality (because of the symmetry of the curves), we can put this point in the first quadrant. According to part (a), the curve or curves of the family passing through this point are identified by the constants C satisfying the quadratic equation:

$$C^2 - (x_0^2 + y_0^2 + 1)C + x_0^2 \;=\; 0.$$

This quadratic always has two distinct positive roots. Hence, there will be two curves of the family passing through (x_0, y_0). Our aim is to show that one is an ellipse, the other is a hyperbola, and the two curves are orthogonal to each other. The two values of C are

$$C_1 \;=\; \frac{1}{2}\Big\{\, x_0^2 + y_0^2 + 1 + \sqrt{\big(x_0^2 + y_0^2 + 1\big)^2 - 4x_0^2}\,\Big\},$$

$$C_2 \;=\; \frac{1}{2}\Big\{\, x_0^2 + y_0^2 + 1 - \sqrt{\big(x_0^2 + y_0^2 + 1\big)^2 - 4x_0^2}\,\Big\}.$$

They satisfy

$$C_1C_2 = x_0^2, \qquad C_1 + C_2 = x_0^2 + y_0^2 + 1, \qquad (C_1 - 1)(C_2 - 1) = -y_0^2.$$

For fixed x_0, the quantity under the square root is minimised by taking $y_0 = 0$. Hence the minimum value of the quantity under the square root is $(x_0^2 - 1)^2$. This gives

$$\begin{aligned} C_1 &> \frac{1}{2}\Big\{\, x_0^2 + 1 + \big|\, x_0^2 - 1 \,\big| \,\Big\} \\ &= \max\big\{x_0^2, 1\big\}. \end{aligned}$$

Hence the curve through (x_0, y_0) with parameter C_1 is an ellipse. But

$$C_1C_2 = x_0^2 \quad \text{implies} \quad 0 < C_2 < \min\big\{1, x_0^2\big\}.$$

Hence the curve with parameter C_2 is a hyperbola. To prove that the ellipse is orthogonal to the hyperbola, we need to show that y' at (x_0, y_0) in one case is minus the reciprocal of y' at (x_0, y_0) in the other case. The quickest way is to use the auxiliary result $C = x/(x + yy')$ in part (a). Using subscripts 1 and 2 to denote the ellipse and hyperbola, respectively, we have

$$(y')_1 = \frac{(1 - C_1)x_0}{C_1 y_0}, \qquad (y')_2 = \frac{(1 - C_2)x_0}{C_2 y_0}.$$

The product of the slopes of the tangents is

$$(y')_1(y')_2 = \frac{(1 - C_1)(1 - C_2)x_0^2}{C_1C_2y_0^2} = \frac{(-y_0^2)x_0^2}{x_0^2y_0^2} = -1.$$

Another way to see the last result is to arrange the differential equation in part (a) as a quadratic equation for y':

$$xy(y')^2 + (x^2 - y^2 - 1)y' - xy = 0.$$

At (x_0, y_0) not on the axes, this gives two real values of y'. The product of the roots is -1.

1. Find the general solution of

(a) $\dfrac{dx}{dt} - tx = t$ (b) $\dfrac{dy}{dx} = \dfrac{4x^3 - y}{x}$

(c) $\dfrac{dy}{dx} + 2y = e^{-x}$ (d) $x^2\dfrac{dy}{dx} + (1-2x)y = x^2$

Solution

(a) The integrating factor is $e^{\int(-t)\,dt} = e^{-t^2/2}$. Multiplying the equation by this integrating factor gives $(d/dt)\{e^{-t^2/2}x\} = te^{-t^2/2}$, and then integration gives $e^{-t^2/2}x = \displaystyle\int te^{-t^2/2}\,dt = -e^{-t^2/2} + C$ and so $x = -1 + Ce^{t^2/2}$.

(b) Rewrite as $y' + x^{-1}y = 4x^2$. The integrating factor is $e^{\int x^{-1}\,dx} = e^{\ln|x|} = |x|$, but x will do. Multiplying the equation by this factor gives $(xy)' = 4x^3$, and then integrating gives $xy = x^4 + C$ and so $y = x^3 + Cx^{-1}$.

(c) The integrating factor is $e^{\int 2\,dx} = e^{2x}$. Multiplying the equation by this factor gives $(d/dx)(e^{2x}y) = e^x$, and then integrating gives $e^{2x}y = e^x + C$ and so $y = e^{-x} + Ce^{-2x}$.

(d) Rewrite as $y' + \left(x^{-2} - 2x^{-1}\right)y = 1$. The integrating factor is $e^{\int\left(x^{-2}-2x^{-1}\right)dx} = e^{-x^{-1}-2\ln|x|} = x^{-2}e^{-1/x}$. Multiplying the equation by this factor gives $(d/dx)(x^{-2}e^{-1/x}y) = x^{-2}e^{-1/x}$, and then integrating gives $x^{-2}e^{-1/x}y = e^{-1/x} + C$ and so $y = x^2 + Cx^2e^{1/x}$.

2. Find the particular solution of

(a) $\dfrac{dy}{dx} + y\tan x = \sec x, \quad y = 2$ when $x = 0$ (b) $\dfrac{dy}{dx} = \dfrac{2y}{x} + x^4, \quad y = 1$ when $x = 1$

(c) $\dfrac{dx}{dt} + 4x = e^{-4t}\sin 2t, \quad x(0) = 1/2$ (d) $(1+x^2)\dfrac{dy}{dx} + 2xy = 4 + 2x, \quad y(0) = 4$

Solution

(a) The integrating factor is $e^{\int\tan x\,dx} = e^{\ln|secx|} = |\sec x|$, but $\sec x$ will do. Multiplying the equation by this factor gives $(d/dx)(y\sec x) = \sec^2 x$, and then integrating gives $y\sec x = \tan x + C$ and so $y = \sin x + C\cos x$. Putting $x = 0$ then gives $y(0) = C$. But we are told that $y(0) = 2$, and so $C = 2$. Hence the particular solution is $y = \sin x + 2\cos x$.

(b) Rewrite as $y' - 2x^{-1}y = x^4$. The integrating factor is $e^{\int(-2/x)\,dx} = e^{-2\ln|x|} = x^{-2}$. Multiplying the equation by this factor gives $(d/dx)(x^{-2}y) = x^2$ and then integrating gives $x^{-2}y = (1/3)x^3 + C$ and so $y = (1/3)x^5 + Cx^2$. Putting $x = 1$ then gives $y(1) = 1/3 + C$. But we are told that $y(1) = 1$, and so $C = 2/3$. Hence the particular solution is $y = (1/3)(x^5 + 2x^2)$.

(c) The integrating factor is $e^{\int 4\,dt} = e^{4t}$. Multiplying the equation by this factor gives $(d/dt)(e^{4t}x) = \sin 2t$ and then integrating gives $e^{4t}x = -(1/2)\cos 2t + C$ and so $x = \{-(1/2)\cos 2t + C\}e^{-4t}$. Putting $t = 0$ gives $x(0) = -1/2 + C$. But we are told that $x(0) = 1/2$, and so $C = 1$. Hence the particular solution is $x = e^{-4t}\{1 - (1/2)\cos 2t\}$.

(d) Rewrite as $y' + 2x(1+x^2)^{-1}y = (4+2x)(1+x^2)^{-1}$. The integrating factor is $e^{\int 2x/(1+x^2)\,dx} = e^{\ln(1+x^2)} = 1 + x^2$. Multiplying the equation by this factor gives $(d/dx)\big((1+x^2)y\big) = 4 + 2x$ and then integrating gives $(1+x^2)y = 4x + x^2 + C$ and so $y = (x^2+4x+C)/(x^2+1)$. Putting $x = 0$ gives $y(0) = C$. But we are told that $y(0) = 4$, and so $C = 4$ and the particular solution is $y = (x+2)^2/(x^2+1)$.

3. The Howard family borrows \$176,000 to buy a house, and plans to make frequent regular repayments of increasing amounts so that the rate of repayment t years after the start of the loan will be $\$R(t)$/year, where $R(t) = R_0\big(1 + (1/80)t^2\big)$ and R_0 is the initial repayment rate. The interest rate is fixed at 5% per annum, and interest charges are added to the loan amount at frequent regular intervals.

(a) Assuming repayments and interest charges are so frequent that they are effectively continuous, show that the loan amount L varies with time according to the differential equation,

$$\frac{dL}{dt} = \frac{L}{20} - R_0\left(1 + \frac{t^2}{80}\right).$$

(b) Solve this equation, and hence obtain an expression for the amount still owed after t years.

(c) Show that the initial repayment rate R_0 must exceed \$800/year or else the debt will eventually grow out of control.

(d) If $R_0 = \$1000$/year, what is the remaining debt after 20 years?

Solution

(a) If the time interval Δt between consecutive repayments is small, then the amount repaid each time will be $R(t) \times \Delta t$. Also the amount of interest charged each time will be $L \times 5/100 \times \Delta t$. Hence, at the end of each such interval, the loan amount will increase by an amount $\Delta L = (L \times 5/100 \times \Delta t) - (R(t) \times \Delta t)$. Dividing this equation by Δt gives

$$\frac{\Delta L}{\Delta t} = \frac{5L}{100} - R(t) = \frac{L}{20} - R_0\left(1 + \frac{t^2}{80}\right).$$

Since Δt is small, this equation is approximated well by the differential equation,

$$\frac{dL}{dt} = \frac{L}{20} - R_0\left(1 + \frac{t^2}{80}\right).$$

(b) Rearranging the DE as $dL/dt - L/20 = -R_0\big(1 + (t^2/80)\big)$, we see that the integrating factor is $e^{\int(-1/20)\,dt} = e^{-t/20}$. Multiplying the DE by this factor gives $(d/dt)\big(e^{-t/20}L\big) = -R_0\big(1 + (t^2/80)\big)e^{-t/20}$. Integrating by parts then gives

$$\begin{aligned}
e^{-t/20}L(t) &= 20R_0 \int \left(1 + \frac{t^2}{80}\right)\frac{d}{dt}\,e^{-t/20}\,dt \\
&= 20R_0\left[\left(1 + \frac{t^2}{80}\right)e^{-t/20} - \int \frac{t}{40}\,e^{-t/20}\,dt\right] \\
&= 20R_0\left[\left(1 + \frac{t^2}{80}\right)e^{-t/20} + 20\int \frac{t}{40}\frac{d}{dt}\,e^{-t/20}\,dt\right] \\
&= 20R_0\left[\left(1 + \frac{t^2}{80}\right)e^{-t/20} + 20\left(\frac{t}{40}\,e^{-t/20} - \int \frac{1}{40}\,e^{-t/20}\,dt\right)\right] \\
&= 20R_0\left[\left(1 + \frac{t^2}{80}\right)e^{-t/20} + 20\left(\frac{t}{40}\,e^{-t/20} + \frac{1}{2}\,e^{-t/20}\right)\right] + C,
\end{aligned}$$

and hence,

$$L(t) = 20R_0\left(11 + \frac{t}{2} + \frac{t^2}{80}\right) + Ce^{t/20},$$

where C is an arbitrary constant of integration. Taking $t = 0$ gives $L(0) = 220R_0 + C$; but we are told that the initial amount of the loan is $L(0) = 176000$, and so we deduce that $C = 176000 - 220R_0$.

Hence the amount (in dollars) owed after t years is

$$L(t) = 20R_0\left(11 + \frac{t}{2} + \frac{t^2}{80}\right) + (176000 - 220R_0)e^{t/20}.$$

(c) If $R_0 < 176000/220 = 800$, then both terms in the above expression are positive and increasing for all values of t; in fact the second term increases exponentially. If $R_0 = 800$, the second term is always zero but the first term increases forever. In either case, the debt will increase without limit. However, if $R_0 > 800$ then the second term is negative, and at some time in the future (i.e., at some positive value of t) it will cancel the first term. At this time $L(t)$ will vanish and the debt will be fully paid off.

4. Use the method of partial fractions to evaluate the following integrals of rational functions:

(a) $\displaystyle\int \frac{x}{(x-1)(x-3)}\,dx, \quad 1 < x < 3,$ (b) $\displaystyle\int \frac{x^5}{(x-1)(x-3)}\,dx, \quad 1 < x < 3,$

(c) $\displaystyle\int \frac{6x^4}{x^6-1}\,dx, \quad x > 1,$ (d) $\displaystyle\int \frac{2x^2}{x^4+1}\,dx.$

Evaluate the following integrals with the help of suitable substitutions:

(e) $\displaystyle\int \sqrt{\tan\theta}\,d\theta, \quad 0 < \theta < \pi/2,$ (f) $\displaystyle\int \frac{\sqrt{1+y^2}}{y^2}\,dy, \quad y > 0,$

(g) $\displaystyle\int \frac{d\theta}{(a+b\cos\theta)^2}, \quad 0 < \theta < \pi, \ |b| < a,$ (h) $\displaystyle\int \frac{1}{x}\left(\frac{x+1}{x-1}\right)^{2/3} dx, \quad x > 1.$

Solution

(a) Let

$$\frac{x}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3}.$$

Then

$$A = \left[\frac{x}{x-3}\right]_{x=1} = -\frac{1}{2}, \qquad B = \left[\frac{x}{x-1}\right]_{x=3} = \frac{3}{2}.$$

Thus

$$\frac{x}{(x-1)(x-3)} = \frac{1}{2}\left\{-\frac{1}{x-1} + \frac{3}{x-3}\right\}.$$

The required indefinite integral on the interval $1 < x < 3$ is

$$\int \frac{x}{(x-1)(x-3)}\,dx = \frac{1}{2}\Big\{3\ln(3-x) - \ln(x-1)\Big\} + K.$$

(b) Long division of polynomials gives

$$\frac{x^5}{(x-1)(x-3)} = x^3 + 4x^2 + 13x + 40 + \frac{121x - 120}{(x-1)(x-3)}.$$

The last term has the partial fraction expansion,

$$\frac{121x-120}{(x-1)(x-3)} = \frac{1}{2}\left\{-\frac{1}{x-1} + \frac{243}{x-3}\right\}.$$

Hence, the integral is

$$\int \frac{x^5}{(x-1)(x-3)}\,dx = \frac{1}{4}x^4 + \frac{4}{3}x^3 + \frac{13}{2}x^2 + 40x + \frac{243}{2}\ln(3-x) - \frac{1}{2}\ln(x-1) + K.$$

(c) The denominator factorises as

$$x^6 - 1 = (x^3 - 1)(x^3 + 1) = (x - 1)(x + 1)(x^2 + x + 1)(x^2 - x + 1).$$

Let

$$\frac{6x^4}{x^6 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{Cx + D}{x^2 + x + 1} + \frac{Ex + F}{x^2 - x + 1}.$$

The evenness of $6x^4/(x^6 - 1)$ implies that $B = -A,\ E = -C$ and $F = D$. So

$$\frac{6x^4}{x^6 - 1} = \frac{A}{x - 1} - \frac{A}{x + 1} + \frac{Cx + D}{x^2 + x + 1} - \frac{Cx - D}{x^2 - x + 1}.$$

We can get A quickly as follows:

$$A = \lim_{x \to 1} \frac{6x^4(x - 1)}{x^6 - 1} = \left[\frac{6x^4}{(x + 1)(x^4 + x^2 + 1)}\right]_{x=1} = 1.$$

Then

$$\begin{aligned}
\frac{Cx + D}{x^2 + x + 1} - \frac{Cx - D}{x^2 - x + 1} &= \frac{6x^4}{x^6 - 1} - \frac{1}{x - 1} + \frac{1}{x + 1} \\
&= \frac{6x^4}{x^6 - 1} - \frac{2}{x^2 - 1} \\
&= \frac{6x^4 - 2(x^4 + x^2 + 1)}{x^6 - 1} \\
&= \frac{4x^4 - 2x^2 - 2}{x^6 - 1} \\
&= \frac{2(x^2 - 1)(2x^2 + 1)}{x^6 - 1} \\
&= \frac{4x^2 + 2}{x^4 + x^2 + 1}.
\end{aligned}$$

Putting $x = 0$ gives $D = 1$. Putting $x = 1$ gives $C = -1$. This completes the partial fraction expansion,

$$\frac{6x^4}{x^6 - 1} = \frac{1}{x - 1} - \frac{1}{x + 1} - \frac{x - 1}{x^2 + x + 1} + \frac{x + 1}{x^2 - x + 1}.$$

To integrate the last two fractions, let $x = y - 1/2$ and $x = y + 1/2$, respectively. Then

$$\begin{aligned}
\int \frac{x - 1}{x^2 + x + 1}\, dx &= \int \frac{y - 3/2}{y^2 + 3/4}\, dy \\
&= \frac{1}{2} \ln\Big(y^2 + \frac{3}{4}\Big) - \frac{3}{2} \cdot \frac{2}{\sqrt{3}} \tan^{-1} \frac{2y}{\sqrt{3}} + K_1 \\
&= \frac{1}{2} \ln(x^2 + x + 1) - \sqrt{3}\, \tan^{-1} \frac{2x + 1}{\sqrt{3}} + K_1,
\end{aligned}$$

and, similarly,

$$\int \frac{x + 1}{x^2 - x + 1}\, dx = \frac{1}{2} \ln(x^2 - x + 1) + \sqrt{3}\, \tan^{-1} \frac{2x - 1}{\sqrt{3}} + K_2.$$

The integral of the given function on the interval $x > 1$ is

$$\begin{aligned}
\int \frac{6x^4}{x^6 - 1}\, dx &= \ln(x - 1) - \ln(x + 1) - \frac{1}{2} \ln(x^2 + x + 1) + \sqrt{3} \tan^{-1} \frac{2x + 1}{\sqrt{3}} \\
&\qquad + \frac{1}{2} \ln(x^2 - x + 1) + \sqrt{3}\, \tan^{-1} \frac{2x - 1}{\sqrt{3}} + K_3 \\
&= -\ln\left(\frac{x + 1}{x - 1}\right) - \frac{1}{2} \ln\left(\frac{x^2 + x + 1}{x^2 - x + 1}\right) + \sqrt{3} \tan^{-1} \frac{2x + 1}{\sqrt{3}} + \sqrt{3}\, \tan^{-1} \frac{2x - 1}{\sqrt{3}} + K_3.
\end{aligned}$$

This result can be written in other ways, for example,

$$\int \frac{6x^4}{x^6-1}\,dx \;=\; \coth^{-1}(x^3) - 3\coth^{-1}x - \sqrt{3}\,\tan^{-1}\!\left(\frac{\sqrt{3}\,x}{x^2-1}\right) + K,$$

where we renamed $K_3 = K - \pi$.

(d) The denominator factorises as

$$x^4+1 \;=\; (x^2+1)^2 - 2x^2 \;=\; (x^2-\sqrt{2}\,x+1)(x^2+\sqrt{2}\,x+1).$$

Let

$$\begin{aligned}\frac{2x^2}{x^4+1} &= \frac{Ax+B}{x^2-\sqrt{2}\,x+1} + \frac{Cx+D}{x^2+\sqrt{2}\,x+1}\\ &= \frac{Ax+B}{x^2-\sqrt{2}\,x+1} - \frac{Ax-B}{x^2+\sqrt{2}\,x+1},\end{aligned}$$

where we used the evenness of the given function $2x^2/(x^4+1)$. Putting $x=0$ gives $B=0$. Then $2x^2 = Ax(x^2+\sqrt{2}\,x+1) - Ax(x^2-\sqrt{2}\,x+1) = 2A\sqrt{2}\,x^2$. This gives $A = 1/\sqrt{2}$. The required partial fraction expansion is

$$\frac{2x^2}{x^4+1} \;=\; \frac{1}{\sqrt{2}}\left\{\frac{x}{x^2-\sqrt{2}\,x+1} - \frac{x}{x^2+\sqrt{2}\,x+1}\right\}.$$

With the substitution $x = y + 1/\sqrt{2}$, we get

$$\begin{aligned}\int \frac{x}{x^2-\sqrt{2}\,x+1}\,dx &= \int \frac{y+1/\sqrt{2}}{y^2+1/2}\,dy\\ &= \frac{1}{2}\ln(y^2+1/2) + \tan^{-1}\sqrt{2}\,y + C_1\\ &= \frac{1}{2}\ln\bigl(x^2-\sqrt{2}\,x+1\bigr) + \tan^{-1}(\sqrt{2}\,x-1) + C_1,\end{aligned}$$

and, similarly,

$$\int \frac{x}{x^2+\sqrt{2}\,x+1}\,dx \;=\; \frac{1}{2}\ln\bigl(x^2+\sqrt{2}\,x+1\bigr) - \tan^{-1}(\sqrt{2}\,x+1) + C_2.$$

Hence, the integral of the given function is

$$\int \frac{2x^2}{x^4+1}\,dx \;=\; -\frac{1}{2\sqrt{2}}\ln\!\left(\frac{x^2+\sqrt{2}\,x+1}{x^2-\sqrt{2}\,x+1}\right) + \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\,x-1) + \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\,x+1) + K.$$

(e) To integrate $\sqrt{\tan\theta}$, the natural substitution to try is $x = \sqrt{\tan\theta}$. Then

$$\theta \;=\; \tan^{-1}(x^2), \qquad d\theta \;=\; \frac{2x}{1+x^4}\,dx.$$

The integral becomes

$$\int \sqrt{\tan\theta}\,d\theta \;=\; \int \frac{2x^2}{1+x^4}\,dx.$$

This is the integral in part (d). In terms of the variable θ, this integral reads

$$\begin{aligned}\int \sqrt{\tan\theta}\,d\theta \;=\; &-\frac{1}{2\sqrt{2}}\ln\!\left(\frac{\tan\theta+\sqrt{2\tan\theta}+1}{\tan\theta-\sqrt{2\tan\theta}+1}\right)\\ &+\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2\tan\theta}-1) + \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2\tan\theta}+1) + K.\end{aligned}$$

This can be rewritten in various ways using trigonometric identities. For example,

$$\int \sqrt{\tan\theta}\,d\theta \;=\; \frac{1}{\sqrt{2}}\ln\bigl(\cos\theta+\sin\theta-\sqrt{\sin 2\theta}\bigr) + \frac{1}{\sqrt{2}}\cos^{-1}(\cos\theta-\sin\theta) + K.$$

(f) To evaluate integrals containing $\sqrt{1+y^2}$, three standard methods are available. Any of these methods could be the fastest in a particular case. In the present problem, the hyperbolic substitution is the best, the trigonometric substitution is not quite as fast, and the direct algebraic substitution is the least efficient.

Method 1. The quickest method is to let $y = \sinh\theta$. Then $\sqrt{1+y^2} = \cosh\theta$ and $dy = \cosh\theta\, d\theta$. The integral becomes

$$\begin{aligned}
\int \frac{\sqrt{1+y^2}}{y^2}\, dy &= \int \frac{\cosh^2\theta}{\sinh^2\theta}\, d\theta \\
&= \int \frac{\sinh^2\theta + 1}{\sinh^2\theta}\, d\theta \\
&= \int \left(1 + \operatorname{cosech}^2\theta\right) d\theta \\
&= \theta - \coth\theta + K \\
&= \sinh^{-1} y - \frac{\sqrt{1+y^2}}{y} + K.
\end{aligned}$$

This result can also be written,

$$\int \frac{\sqrt{1+y^2}}{y^2}\, dy = \ln\left(y + \sqrt{1+y^2}\,\right) - \frac{\sqrt{1+y^2}}{y} + K.$$

Method 2. An alternative substitution is $y = \tan\theta$. Then $\sqrt{1+y^2} = \sec\theta$ and $dy = \sec^2\theta\, d\theta$. The integral becomes

$$\begin{aligned}
\int \frac{\sqrt{1+y^2}}{y^2}\, dy &= \int \frac{\sec^3\theta}{\tan^2\theta}\, d\theta \\
&= \int \frac{d\theta}{\sin^2\theta\cos\theta} \\
&= \int \frac{\cos\theta\, d\theta}{\sin^2\theta\cos^2\theta} \\
&= \int \frac{dx}{x^2(1-x^2)} \qquad\qquad (x = \sin\theta) \\
&= \int \left\{ \frac{1}{x^2} + \frac{1}{1-x^2} \right\} dx \\
&= -\frac{1}{x} + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + K \\
&= \frac{1}{2}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right) - \frac{1}{\sin\theta} + K \\
&= \frac{1}{2}\ln\left(\frac{\sec\theta+\tan\theta}{\sec\theta-\tan\theta}\right) - \frac{\sec\theta}{\tan\theta} + K \\
&= \ln\left(\sec\theta + \tan\theta\right) - \frac{\sec\theta}{\tan\theta} + K \\
&= \ln\left(\sqrt{1+y^2} + y\right) - \frac{\sqrt{1+y^2}}{y} + K,
\end{aligned}$$

in agreement with the first method. We see that the trigonometric substitution is not as fast as the hyperbolic substitution in this particular example. Notice that we did not complete the partial fraction expansion on the fifth line because it is convenient to regard the following as a standard integral:

$$\int \frac{dx}{a^2 - x^2} = \begin{cases} \dfrac{1}{2a}\ln\left(\dfrac{a+x}{a-x}\right) + C = \dfrac{1}{a}\tanh^{-1}\left(\dfrac{x}{a}\right) + C, & -a < x < a, \\ \dfrac{1}{2a}\ln\left(\dfrac{x+a}{x-a}\right) + C = \dfrac{1}{a}\coth^{-1}\left(\dfrac{x}{a}\right) + C, & x > a > 0. \end{cases}$$

The substitution $x = \sin\theta$ can be avoided by the following intermediate steps:

$$\begin{aligned} \int \frac{d\theta}{\sin^2\theta\cos\theta} &= \int \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta\cos\theta}\, d\theta \\ &= \int \big(\sec\theta + \operatorname{cosec}\theta\cot\theta\big)\, d\theta \\ &= \ln\big(\sec\theta + \tan\theta\big) - \operatorname{cosec}\theta + K. \end{aligned}$$

This agrees with the corresponding stage of the preceding calculation, and one concludes as before.

Method 3. The direct algebraic substitution is

$$x = y + \sqrt{1+y^2}, \qquad y = \frac{x^2-1}{2x},$$

$$\sqrt{1+y^2} = \frac{x^2+1}{2x}, \qquad dy = \frac{x^2+1}{2x^2}\, dx, \qquad x > 1.$$

The integral becomes

$$\begin{aligned} \int \frac{\sqrt{1+y^2}}{y^2}\, dy &= \int \frac{(x^2+1)^2}{x(x^2-1)^2}\, dx \\ &= \frac{1}{2}\int \frac{(u+1)^2}{u(u-1)^2}\, du \qquad\qquad (u = x^2), \end{aligned}$$

where we took advantage of the odd integrand in x to get a simpler rational integrand in u. Let

$$\frac{(u+1)^2}{u(u-1)^2} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{(u-1)^2}.$$

Then

$$A = \left[\frac{(u+1)^2}{(u-1)^2}\right]_{u=0} = 1, \qquad C = \left[\frac{(u+1)^2}{u}\right]_{u=1} = 4.$$

To get B, multiply both sides by u and let $u \to \infty$. This gives $A + B = 1$, which implies that $B = 0$. The required partial fraction expansion and its integral on the domain $u > 1$ are

$$\frac{(u+1)^2}{u(u-1)^2} = \frac{1}{u} + \frac{4}{(u-1)^2},$$

$$\int \frac{(u+1)^2}{u(u-1)^2}\, du = \ln u - \frac{4}{u-1} + K_1.$$

In terms of the original variable y, we get

$$\begin{aligned} \int \frac{\sqrt{1+y^2}}{y^2}\, dy &= \frac{1}{2}\left(\ln u - \frac{4}{u-1} + K_1\right) \\ &= \ln x - \frac{2}{x^2-1} + K_2 \\ &= \ln\big(y + \sqrt{1+y^2}\,\big) - \frac{1}{y^2 + y\sqrt{1+y^2}} + K_2 \\ &= \ln\big(y + \sqrt{1+y^2}\,\big) - \frac{\sqrt{1+y^2} - y}{y} + K_2 \\ &= \ln\big(y + \sqrt{1+y^2}\,\big) - \frac{\sqrt{1+y^2}}{y} + K, \end{aligned}$$

where we renamed the constant as $K_1 = 2K_2$ and $K_2 = K - 1$. This result is in agreement with the first two methods.

(g) To evaluate integrals of rational functions of $\cos\theta$ and $\sin\theta$, the standard change of variable is $t = \tan(\theta/2)$. (In cases where the integrand is built out of $\cos^2\theta$, $\sin^2\theta$ and $\cos\theta\sin\theta$, a better substitution is $t = \tan\theta$, but that is not applicable here.) With $t = \tan(\theta/2)$, we have

$$\cos\theta = \frac{1-t^2}{1+t^2}\,, \qquad \sin\theta = \frac{2t}{1+t^2}\,, \qquad d\theta = \frac{2\,dt}{1+t^2}\,.$$

The integral becomes

$$\int \frac{1}{(a+b\cos\theta)^2}\,d\theta = \int \frac{2(1+t^2)}{\{a+b+(a-b)t^2\}^2}\,dt, \qquad |b| < a.$$

We have arrived at a partial fraction problem of the fourth type. Let c^2 denote $(a+b)/(a-b)$ and define

$$I_1 = \int \frac{dt}{(c^2+t^2)^2}\,, \qquad I_2 = \int \frac{t^2\,dt}{(c^2+t^2)^2}\,.$$

Then

$$c^2 I_1 + I_2 = \int \frac{dt}{c^2+t^2} = \frac{1}{c}\tan^{-1}\Big(\frac{t}{c}\Big) + \text{const.}$$

The second integral I_2 is better adapted to integration by parts. Let $U = t$ and $V' = t/(c^2+t^2)^2$. Then $U' = 1$ and $V = -1/(2(c^2+t^2))$. We get

$$\begin{aligned}
I_2 &= -\frac{t}{2(c^2+t^2)} + \frac{1}{2}\int \frac{dt}{c^2+t^2} \\
&= \frac{1}{2c}\tan^{-1}\Big(\frac{t}{c}\Big) - \frac{t}{2(c^2+t^2)} + \text{const.}, \\
I_1 &= \frac{1}{c^2}\Big\{\frac{1}{c}\tan^{-1}\Big(\frac{t}{c}\Big) - I_2\Big\} + \text{const.} \\
&= \frac{1}{2c^3}\tan^{-1}\Big(\frac{t}{c}\Big) + \frac{t}{2c^2(c^2+t^2)} + \text{const.}
\end{aligned}$$

With $c^2 = (a+b)/(a-b)$, these integrals become

$$\begin{aligned}
\int \frac{dt}{\{a+b+(a-b)t^2\}^2} &= \frac{1}{2(a+b)^{3/2}\sqrt{a-b}}\tan^{-1}\Big(t\sqrt{\frac{a-b}{a+b}}\Big) \\
&\qquad + \frac{t}{2(a+b)\{a+b+(a-b)t^2\}} + \text{const.}, \\
\int \frac{t^2\,dt}{\{a+b+(a-b)t^2\}^2} &= \frac{1}{2(a-b)^{3/2}\sqrt{a+b}}\tan^{-1}\Big(t\sqrt{\frac{a-b}{a+b}}\Big) \\
&\qquad - \frac{t}{2(a-b)\{a+b+(a-b)t^2\}} + \text{const.}
\end{aligned}$$

The required integral in the variable t is

$$\begin{aligned}
\int \frac{2(1+t^2)\,dt}{\{a+b+(a-b)t^2\}^2} &= \Big(\frac{1}{a+b} + \frac{1}{a-b}\Big)\frac{1}{\sqrt{a^2-b^2}}\tan^{-1}\Big(t\sqrt{\frac{a-b}{a+b}}\Big) \\
&\qquad + \Big(\frac{1}{a+b} - \frac{1}{a-b}\Big)\frac{t}{a+b+(a-b)t^2} + \text{const.} \\
&= \frac{2a}{(a^2-b^2)^{3/2}}\tan^{-1}\Big(t\sqrt{\frac{a-b}{a+b}}\Big) \\
&\qquad - \frac{2bt}{(a^2-b^2)\{a+b+(a-b)t^2\}} + K.
\end{aligned}$$

In the original variable θ, we do not want an answer containing half-angle expressions. Let

$$\phi = \tan^{-1}\left(t\sqrt{\frac{a-b}{a+b}}\right) = \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\left(\frac{\theta}{2}\right)\right).$$

Then

$$\begin{aligned}\cos 2\phi &= \cos^2\phi - \sin^2\phi \\ &= \frac{1-\tan^2\phi}{1+\tan^2\phi} \\ &= \frac{a+b-(a-b)\tan^2(\theta/2)}{a+b+(a-b)\tan^2(\theta/2)} \\ &= \frac{(a+b)\cos^2(\theta/2)-(a-b)\sin^2(\theta/2)}{(a+b)\cos^2(\theta/2)+(a-b)\sin^2(\theta/2)} \\ &= \frac{b+a\cos\theta}{a+b\cos\theta},\end{aligned}$$

and

$$\begin{aligned}\frac{2t}{a+b+(a-b)t^2} &= \frac{2\sin(\theta/2)\cos(\theta/2)}{(a+b)\cos^2(\theta/2)+(a-b)\sin^2(\theta/2)} \\ &= \frac{\sin\theta}{a+b\cos\theta}.\end{aligned}$$

The final form of the integral is

$$\int\frac{d\theta}{(a+b\cos\theta)^2} = \frac{a}{(a^2-b^2)^{3/2}}\cos^{-1}\left(\frac{b+a\cos\theta}{a+b\cos\theta}\right) - \frac{b\sin\theta}{(a^2-b^2)(a+b\cos\theta)} + K.$$

(h) If a function being integrated contains a fractional power of $(ax+b)/(cx+d)$, then a natural first step is to make the substitution $y=(ax+b)/(cx+d)$. In the present problem, let

$$y = \frac{x+1}{x-1}, \qquad x = \frac{y+1}{y-1}, \qquad dx = -\frac{2\,dy}{(y-1)^2}.$$

Then let $y=u^3$, $dy = 3u^2\,du$. The integral becomes

$$\begin{aligned}\int\frac{1}{x}\left(\frac{x+1}{x-1}\right)^{2/3}dx &= -2\int\frac{y^{2/3}}{y^2-1}\,dy \\ &= -6\int\frac{u^4}{u^6-1}\,du.\end{aligned}$$

This is the integral in part (c). From the result of part (c), we get

$$\begin{aligned}\int\frac{1}{x}\left(\frac{x+1}{x-1}\right)^{2/3}dx &= 3\coth^{-1}u - \coth^{-1}(u^3) + \sqrt{3}\,\tan^{-1}\left(\frac{\sqrt{3}\,u}{u^2-1}\right) + K \\ &= 3\coth^{-1}\left(\left(\frac{x+1}{x-1}\right)^{1/3}\right) - \coth^{-1}\left(\frac{x+1}{x-1}\right) \\ &\quad + \sqrt{3}\,\tan^{-1}\left(\frac{\sqrt{3}\,(x^2-1)^{1/3}}{(x+1)^{2/3}-(x-1)^{2/3}}\right) + K \\ &= 3\tanh^{-1}\left(\left(\frac{x-1}{x+1}\right)^{1/3}\right) - \frac{\ln x}{2} \\ &\quad + \sqrt{3}\,\tan^{-1}\left(\frac{\sqrt{3}\,(x^2-1)^{1/3}}{(x+1)^{2/3}-(x-1)^{2/3}}\right) + K.\end{aligned}$$