Semester 2 **Tutorial Solutions Week 11** 2009

1. (a) Find the general solution for each of the following 2nd-order linear homogeneous DEs:

$$(i)\quad \frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = 0 \quad (ii)\quad \frac{d^2y}{dt^2} + 16y = 0 \qquad (iii)\quad \frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 13y = 0.$$

(b) For each of these DEs, find the particular solution satisfying $y(0) = \dot{y}(0) = -1$.

Solution

(a) (i) The auxiliary equation $m^2 - m - 6 = 0$ has roots $m = 3, -2$, and so the general solution is $y = Ae^{3t} + Be^{-2t}$.

(ii) The auxiliary equation $m^2 + 16 = 0$ has complex roots $m = \pm 4i$, and so the general solution is $y = C\cos 4t + D\sin 4t$.

(iii) The auxiliary equation $m^2 + 6m + 13 = 0$ has complex roots $m = -3 \pm 2i$, and so the general solution is $y = e^{-3t}(C\cos 2t + D\sin 2t)$.

(b) (i) Differentiating the general solution, we see that $\dot{y} = 3Ae^{3t} - 2Be^{-2t}$, and so $\dot{y}(0) = 3A - 2B$. The general solution also gives $y(0) = A + B$. Hence the initial conditions imply $A + B = -1$ and $3A - 2B = -1$, and therefore $A = -3/5$ and $B = -2/5$. So the particular solution which satisfies the DE is $y = -(1/5)(3e^{3t} + 2e^{-2t})$.

(ii) Differentiating the general solution, we see that $\dot{y} = -4C\sin 4t + 4D\cos 4t$, and so $\dot{y}(0) = 4D$. The general solution also gives $y(0) = C$. Hence the initial conditions imply $C = -1$ and $4D = -1$, and therefore $D = -1/4$. So the particular solution which satisfies the DE is $y = -\cos 4t - (1/4)\sin 4t$.

(iii) Differentiating the general solution, we see that
$\dot{y} = (2D - 3C)e^{-3t}\cos 2t - (2C + 3D)e^{-3t}\sin 2t$
and so $\dot{y}(0) = 2D - 3C$. The general solution also gives $y(0) = C$. Hence the initial conditions imply $2D - 3C = -1$ and $C = -1$, and therefore $D = -2$. So the particular solution which satisfies the DE is $y = -e^{-3t}(\cos 2t + 2\sin 2t)$.

2. Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0,$$

expressing your answer in terms of sine and cosine functions. Find the particular solution satisfying the boundary values $y(0) = 1$ and $y(\pi/4) = 2$. Is there a solution that satisfies the boundary values $y(0) = 1$ and $y(\pi) = 1$?

Solution

The auxiliary equation is $m^2 - 2m + 5 = 0$, which has roots $m = 1 \pm 2i$, and so the general solution is $y = e^t(E\cos 2t + F\sin 2t)$.

Hence $y(0) = E$ and $y(\pi/4) = e^{\pi/4}F$. So if $y(0) = 1$ and $y(\pi/4) = 2$ then $E = 1$ and $F = 2e^{-\pi/4}$, and hence the particular solution is

$$y = e^t\big(\cos 2t + 2e^{-\pi/4}\sin 2t\big).$$

The second set of boundary values leads to the equations $E = 1$ and $e^{\pi}E = 1$, which do not have a solution. Boundary value problems are not always solvable.

3. (From 1995 exam)

(a) Classify each of the following DEs as separable or linear, and find its general solution:

$$(i)\quad \frac{dy}{dx} = \frac{2xy^2}{1+x^2} \qquad (ii)\quad \frac{dy}{dx} = x + \frac{y}{x} \qquad (iii)\quad (x^2+1)\frac{dy}{dx} + x = xy^2$$

(b) Find the particular solution of (iii) for which $y = 0$ when $x = 1$.

(c) Use the substitution $v = xw$ to find the general solution of the differential equation

$$2xv\frac{dv}{dx} = 3v^2 - 4x^2.$$

Solution

(a) (i) Separable:

$$\int y^{-2}\,dy = \int 2x(1+x^2)^{-1}\,dx \Rightarrow -y^{-1} = \ln(1+x^2) + C.$$

Hence the general solution is

$$y = -1/\{\ln(1+x^2) + C\}.$$

(ii) Linear:

$$\frac{dy}{dx} - \frac{1}{x}y = x \Rightarrow \text{I.F.} = \exp\Big(\int -\frac{1}{x}\,dx\Big) = \exp(-\ln|x|) = |x|^{-1}.$$

In fact $1/x$ will do, the absolute value signs being inconsequential.

So $(d/dx)(y/x) = 1 \Rightarrow y = x\int 1\,dx = x(x+C)$.

Hence the general solution is

$$y = x(x+C).$$

(iii) Separable:

The DE can be rewritten as $\dfrac{dy}{y^2-1} = \dfrac{x\,dx}{x^2+1}$.

Integrating gives $(1/2)\ln\big|(y-1)/(y+1)\big| = (1/2)\ln(x^2+1) + C$.

Hence, $(y-1)/(y+1) = \pm e^{2C}(x^2+1) = A(x^2+1)$.

Solving for y gives the general solution,

$$y = \frac{1 + A(x^2+1)}{1 - A(x^2+1)}.$$

(b) Putting $y = 0$ when $x = 1$ in the general solution gives $A = -1/2$. The particular solution is therefore $y = (1-x^2)/(3+x^2)$.

(c) We now find that $\dfrac{dv}{dx} = \dfrac{d}{dx}(xw) = w + x\dfrac{dw}{dx}$, so the DE becomes

$$\begin{aligned}
2x(xw)\left(w + x\frac{dw}{dx}\right) &= 3(xw)^2 - 4x^2 \\
\Rightarrow 2x^2w^2 + 2x^3w\frac{dw}{dx} &= 3x^2w^2 - 4x^2 \\
\Rightarrow 2xw\frac{dw}{dx} &= w^2 - 4.
\end{aligned}$$

Separating and integrating, we get

$$\begin{aligned}
\int \frac{2w}{w^2-4}\,dw &= \int \frac{1}{x}\,dx \\
\Rightarrow \ln|w^2 - 4| &= \ln|x| + C \\
\Rightarrow w^2 - 4 &= \pm e^C x = Ax \\
\Rightarrow w &= \pm\sqrt{4 + Ax} \\
\Rightarrow v &= \pm x\sqrt{4 + Ax}\,.
\end{aligned}$$

4. Solve the following differential equations by making suitable substitutions:

(a) $\dfrac{dy}{dx} = \dfrac{y^3}{x(x^2+y^2)}$ (b) $\dfrac{dy}{dx} = \dfrac{1}{x+y}$

(c) $\dfrac{dy}{dx} = \dfrac{x+y}{x+y+2}$ (d) $\dfrac{dy}{dx} = \dfrac{1}{(x+2y)^2+1}$

Solution

(a) This equation is homogeneous. Define $v = y/x$. Then $y' = (xv)' = v + xv'$, and so the DE implies $v + xv' = v^3/(1+v^2)$ or $xv' = -v/(1+v^2)$. This is separable; we get

$$\begin{aligned}
-\int \frac{1+v^2}{v}\,dv &= \int \frac{dx}{x} \\
\Rightarrow -\ln|v| - \frac{1}{2}v^2 &= \ln|x| + C \\
\Rightarrow e^{-v^2/2} &= Axv = Ay.
\end{aligned}$$

Then recalling that $v = y/x$ gives $e^{y^2/(2x^2)} = Ay$. It is not possible to write y explicitly as a function of x.

(b) Define $w = x + y$. Then $w' = 1 + y' = 1 + (1/w) = (w+1)/w$, which is separable; we get

$$\begin{aligned}
\int \frac{w\,dw}{w+1} &= \int dx \\
\Rightarrow w - \ln|w+1| &= x + C \\
\Rightarrow \frac{e^w}{w+1} &= Ae^x \\
\Rightarrow \frac{e^{x+y}}{x+y+1} &= Ae^x \\
\Rightarrow e^y &= A(x+y+1).
\end{aligned}$$

It is not possible to write y explicitly as a function of x. [Another way to solve this equation is to take reciprocals and get a linear equation in x.]

(c) Define $w = x + y$. Then $w' = 1 + y' = 1 + w/(w + 2) = (2w + 2)/(w + 2)$, which is separable. We get

$$\begin{aligned} \int \frac{(w+2)\,dw}{2w+2} &= \int dx \\ \Rightarrow \frac{w}{2} + \frac{1}{2}\ln|w+1| &= x + C \\ \Rightarrow (w+1)e^{w} &= Ae^{2x} \\ \Rightarrow x + y + 1 &= Ae^{x-y}. \end{aligned}$$

It is not possible to write y explicitly as a function of x.

(d) Define $z = x + 2y$. Then $z' = 1 + 2y' = 1 + 2/(z^2 + 1) = (z^2 + 3)/(z^2 + 1)$, which is separable. We get

$$\begin{aligned} \int \frac{z^2+1}{z^2+3}\,dz &= \int dx \\ \Rightarrow \int \left(1 - \frac{2}{x^2+3}\right) dz &= x + C \\ \Rightarrow z - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{z}{\sqrt{3}}\right) &= x + C \\ \Rightarrow x + 2y - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}}\right) &= x + C \\ \Rightarrow y &= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}}\right) + A. \end{aligned}$$

where A is an arbitrary constant of integration. It is not possible to write y explicitly as a function of x.

5. Solve the following differential equations by the methods indicated:

(a) $3x\dfrac{dy}{dx} + y + x^2y^4 = 0$ (a Bernoulli equation, let $w = 1/y^3$).

(b) $\dfrac{dy}{dx} + xy^2 + 3/(4x^3) = 0$ (a Riccati equation, let $y = 1/(2x^2) + 1/w$).

(c) $x\dfrac{dy}{dx} - y = \dfrac{1}{4}\left(\dfrac{dy}{dx}\right)^4$ (a Clairaut equation, differentiate both sides).

Solution

(a) $w = 1/y^3$ implies $y = w^{-1/3}$ and $y' = -(1/3)w^{-4/3}w'$. Substitution into the Bernoulli equation and rearanging gives the linear equation,

$$x\frac{dw}{dx} - w = x^2.$$

Dividing by x puts it in standard form, after which its integrating factor is seen to be $1/x$. Hence $(d/dx)(w/x) = 1$, which implies the general solution, $w = x(x + C)$. In terms of the original variable y, the solution becomes

$$y = x^{-1/3}(x + C)^{-1/3}.$$

(b) $y = 1/(2x^2) + 1/w$ implies $y' = -1/x^3 - w'/w^2$. Substitution into the Riccati equation and simplifying the result gives the equation $xw' - w = x^2$ that we met in part (a). Hence $w = x(x + C)$ and so

$$y = \frac{3x + C}{2x^2(x + C)}.$$

(c) Differentiate both sides of the Clairaut equation $xy' - y = (y')^4/4$ to get

$$\{(y')^3 - x\}y'' = 0.$$

The two factors give solutions of distinct character. The second factor integrates to $y = Cx + C_2$, with two constants. Substitution into the original equation identifies $C_2 = -C^4/4$. Hence the general solution is

$$y = Cx - \frac{1}{4}C^4.$$

This is a one parameter family of straight lines. There is a curved envelope which is tangent to all the lines. This curve also satisfies the Clairaut equation and is the singular integral. It comes from the factor $(y')^3 - x$. Setting this factor to zero gives $y' = x^{1/3}$. Thence, without further integration, we get

$$y = xy' - (y')^4/4 = x^{4/3} - x^{4/3}/4 = 3x^{4/3}/4.$$

Hence the singular integral is

$$y = \frac{3}{4}x^{4/3}.$$

The straight line solution $y = Cx - (1/4)C^4$ touches this curved solution tangentially at the point $(C^3, 3C^4/4)$.

Remark. Linear differential equations never have singular integrals. Nor do nonlinear differential equations in which the highest derivative appears linearly. Suppose $F(x, y, y') = 0$ is a first-order differential equation for y in which the left-hand side is a polynomial in y and y', the degree in y' being at least 2. A singular integral, if one exists, is a function $y = f(x)$ that satisfies both $F = 0$ and $\partial F/\partial y' = 0$. The curve with equation $y = f(x)$ is tangent to each of the curves in the general solution (a one-parameter family depending on a constant C). We call such a curve an *envelope* of the one-parameter family of curves. In the Clairaut equation above, as in most cases, the singular integral is not a member of the family of solutions forming the general solution. Some people assume this is always true. However, in particular cases, it is possible for a singular integral to be a member of the general solution, just as it is possible for a one-parameter family of curves to contain its own envelope.

6. A tank A contains 100 litres of water in which 5 kilograms of salt has been dissolved. Fresh water enters it at a rate of 2 litres/minute and the resulting mixture, assumed uniform, flows at the same rate into a second tank B which initially contains 50 litres of fresh water. This mixture, also kept uniform, leaves B at the rate of 2 litres/minute. Find the amount of salt in tank B after 50 minutes.

Solution

The mass x of salt in tank A decreases at the rate rc_A, where $r = 2$ litres/minute is the flow rate, and c_A is the concentration of salt in the tank. Thus

$$\frac{dx}{dt} = -rc_A = -r\frac{x}{L_A},$$

where $L_A = 100$ litres is the (constant) volume of tank A. Solving this equation gives $x(t) = x_0 e^{-rt/L_A}$ where $x_0 = 5$ kg is the initial mass of salt in the tank. Dividing by L_A, we see that the salt concentration in the outflow is

$$c_A(t) = \frac{x_0}{L_A} e^{-rt/L_A}.$$

Let y be the mass and $c_B = y/L_B$ be the concentration of salt in tank B, which has volume $V_B = 50$ litres. The rate (kg/minute) at which salt flows into tank B is rc_A, while the rate at which salt flows out is rc_B. Hence the rate of change of the mass of salt in tank B is

$$\frac{dy}{dt} = rc_A - rc_B = \frac{rx_0}{L_A} e^{-rt/L_A} - \frac{ry}{L_B}.$$

This equation is linear, and can be rewritten as

$$e^{-rt/L_B} \frac{d}{dt}\left(e^{rt/L_B} y\right) = \frac{dy}{dt} + \frac{ry}{L_B} = \frac{rx_0}{L_A} e^{-rt/L_A}.$$

Mulitiplying by the integrating factor e^{rt/L_B} and integrating, we get

$$e^{rt/L_B} y = \int \frac{rx_0}{L_A} \exp\left(\frac{rt}{L_B} - \frac{rt}{L_A}\right) dt = \frac{x_0 L_B}{(L_A - L_B)} \exp\left(\frac{rt}{L_B} - \frac{rt}{L_A}\right) + C.$$

When $t = 0$, the right-hand side is just $C + x_0 L_B/(L_A - L_B)$, and the left-hand side is 0 (as the salt concentration in tank B is initially zero). Hence $C = -x_0 L_B/(L_A - L_B)$, and substituting this into the equation above gives

$$y(t) = \frac{x_0 L_B}{(L_A - L_B)} \left[e^{-rt/L_A} - e^{-rt/L_B}\right].$$

Taking $x_0 = 5$ kg, $L_A = 100$ litres, $L_B = 50$ litres, and $r = 2$ litres/minute gives

$$y = 5\left(e^{-t/50} - e^{-t/25}\right).$$

At $t = 50$ minutes, the amount of salt in tank B is

$$y(50) = 5(e - 1)/e^2 = 1.163 \text{ kg}.$$

THE UNIVERSITY OF SYDNEY

MATH1903 INTEGRAL CALCULUS AND MODELLING (ADVANCED)

Semester 2 **Further Exercises Solutions Week 11** 2009

1. Solve the following equations, giving the general solution and then the particular solution $y(x)$ satisfying the given boundary or initial conditions.

(a) $y'' + 4y' + 5y = 0, \qquad y(0) = 2,\ y'(0) = 4$

(b) $y'' - 2y' + y = 0, \qquad y(2) = 0,\ y'(2) = 1$

(c) $2y'' - 7y' + 5y = 0, \qquad y(0) = 1,\ y'(0) = 1$

(d) $y'' + 4y' + 3y = 0, \qquad y(-2) = 1,\ y(2) = 1$

(e) $2y'' - 2y' + 5y = 0, \qquad y(0) = 0,\ y(2) = 2$

(f) $y'' - 4y' + 4y = 0, \qquad y(0) = -2,\ y(1) = 0$

Solution

(a) The auxiliary equation $m^2+4m+5 = 0$ has roots $-2\pm i$, and so the general solution is $y(x) = e^{-2x}(C\cos x+D\sin x)$, which gives $y'(x) = e^{-2x}\{(D-2C)\cos x-(C+2D)\sin x\}$. Hence $y(0) = C$ and $y'(0) = D-2C$, so the initial conditions imply $C = 2$ and $D = 8$, and the particular solution is $y(x) = 2e^{-2x}(\cos x + 4\sin x)$.

(b) The auxiliary equation $m^2 - 2m + 1 = 0$ has one double root $m = 1$, and so the general solution is $y(x) = (A + Bx)e^x$, which gives $y'(x) = (A + B + Bx)e^x$. Hence $y(2) = (A+2B)e^2$ and $y'(2) = (A+3B)e^2$, so the initial conditions imply $A = -2e^{-2}$ and $B = e^{-2}$, and the particular solution is $y(x) = (x-2)e^{x-2}$.

(c) The auxiliary equation $2m^2 - 7m + 5 = 0$ has roots $5/2$ and 1, and so the general solution is $y(x) = Ae^{5x/2} + Be^x$, which gives $y'(x) = (5A/2)e^{5x/2} + Be^x$. Hence $y(0) = A + B$ and $y'(0) = (5A/2) + B$, so the initial conditions imply $A = 0$ and $B = 1$, and the particular solution is $y(x) = e^x$.

(d) The auxiliary equation $m^2 + 4m + 3 = 0$ has roots -1 and -3, and so the general solution is $y(x) = Ae^{-x} + Be^{-3x}$. Hence $y(-2) = Ae^2 + Be^6$ and $y(2) = Ae^{-2} + Be^{-6}$, so the boundary conditions imply $Ae^2 + Be^6 = 1$ and $Ae^{-2} + Be^{-6} = 1$. Solving these simultaneous equations gives

$$A = \frac{\sinh 6}{\sinh 4} = 7.3915, \qquad B = -\frac{\sinh 2}{\sinh 4} = -0.1329,$$

and so the particular solution satisfying the boundary conditions is

$$y(x) = 7.3915\, e^{-x} - 0.1329\, e^{-3x}.$$

(e) The auxiliary equation $2m^2 - 2m + 5 = 0$ has roots $(1\pm 3i)/2$, and so the general solution is $y(x) = e^{x/2}\{A\cos(3x/2) + B\sin(3x/2)\}$. Hence $y(0) = A$, and the first boundary condition implies $A = 0$. Thus $y(2) = Be\sin 3$, and so the second boundary condition implies $B = 2/(e\sin 3) = 5.2137$, and hence the particular solution satisfying the boundary conditions is $y(x) = 5.2137\, e^{x/2}\sin(3x/2)$.

(f) The auxiliary equation $m^2-4m+4 = 0$ has one double root $m = 2$, and so the general solution is $y(x) = (A + Bx)e^{2x}$. Hence $y(0) = A$ and the first boundary condition implies $A = -2$. Thus $y(1) = (-2 + B)e^2$, and so the second boundary condition implies $B = 2$, and hence the particular solution satisfying the boundary conditions is $y(x) = 2(x-1)e^{2x}$.

2. Classify the following DEs as separable or linear, and in each case find the general solution for y as an explicit function of x (from 1991/2/3/4 exams):

(a) $\dfrac{dy}{dx} = y^2 \sin x$

(b) $x\dfrac{dy}{dx} + (1-x)y = e^x$

(c) $(1+x^2)\dfrac{dy}{dx} + 4xy = 2x + 2xy^2$

(d) $y\dfrac{dy}{dx} = e^{-y^2} x \sin x$

(e) $x^3\dfrac{dy}{dx} = -3x^2y + \cot x$

(f) $\dfrac{dy}{dx} + 2xy - e^{-x^2}\cot x = 0.$

(g) $e^{-x}\dfrac{dy}{dx} - xy^2 + 3xy = 2x$

(h) $\dfrac{dy}{dx} = \dfrac{5-2y}{1+x^2}$

Solution

(a) Separable;
$\displaystyle\int y^{-2}\,dy = \int \sin x\ dx \Rightarrow -y^{-1} = -\cos x + C \Rightarrow y = 1/(\cos x - C).$

(b) Linear;
The standard form is $y' + (x^{-1}-1)y = x^{-1}e^x$. The integrating factor is $e^{\int (x^{-1}-1)\,dx} = e^{\ln|x| - x} = |x|e^{-x}$ but $I(x) = xe^{-x}$ will do. Multiplying the DE by $I(x)$ gives $(d/dx)(xe^{-x}y) = 1$ and then integration gives $xe^{-x}y = x + C$. Hence the general solution is $y = (1 + C/x)e^x$.

(c) Separable, as can be seen after rewriting as $(1+x^2)\dfrac{dy}{dx} = 2x(y-1)^2$.
Then get

$$\int \frac{dy}{(y-1)^2}\,dy = \int \frac{2x\,dx}{1+x^2}$$
$$\Rightarrow -\frac{1}{y-1} = \ln(1+x^2) + C$$
$$\Rightarrow y = 1 - \frac{1}{\ln(1+x^2)+C}.$$

(d) Separable;
$\displaystyle\int ye^{y^2}\,dy = \int x\sin x\ dx \Rightarrow \frac{1}{2}e^{y^2} = \sin x - x\cos x + C$ and hence

$$y = \sqrt{\ln(2C + 2\sin x - 2x\cos x)}.$$

(e) Linear, as can be seen by rewriting DE as $y' + 3x^{-1}y = x^{-3}\cot x$.
The integrating factor is $e^{\int 3x^{-1}\,dx} = e^{3\ln|x|} = |x|^3$, but $I(x) = x^3$ will do. Multiplying the DE by $I(x)$ gives $(d/dx)(x^3y) = \cot y$, and then integration gives $x^3y = \ln|\sin x| + C$, and hence $y = x^{-3}\{\ln|\sin x| + C\}$.

(f) Linear.
The integrating factor is $I(x) = e^{\int 2x\,dx} = e^{x^2}$. Multiplying the DE by $I(x)$ gives $(d/dx)(e^{x^2}y) - \cot x = 0$. Integration then gives $e^{x^2}y - \ln|\sin x| = C \Rightarrow y = e^{-x^2}(C + \ln|\sin x|)$.

(g) Separable, as can be seen by rewriting as $e^{-x}\dfrac{dy}{dx} = x(y-2)(y-1)$.

Then get $\displaystyle\int (y-2)^{-1}(y-1)^{-1}\,dy = \int xe^x\,dx \Rightarrow \ln|(y-2)/(y-1)| = (x-1)e^x + C$, and hence

$$\frac{y-2}{y-1} = Ae^{(x-1)e^x} \Rightarrow y = \frac{Ae^{(x-1)e^x}-2}{Ae^{(x-1)e^x}-1}.$$

(h) Separable **and** Linear. Easiest to separate:

$$\begin{aligned} \int (5-2y)^{-1}\,dy &= \int (1+x^2)^{-1}\,dx \\ \Rightarrow -\frac{1}{2}\ln|5-2y| &= \tan^{-1}x + C \\ \Rightarrow \ln|5-2y| &= -2C - 2\tan^{-1}x \\ \Rightarrow 2y-5 &= \pm e^{-2C}\exp(-2\tan^{-1}x) \\ \Rightarrow y &= \frac{5}{2} + A\exp(-2\tan^{-1}x). \end{aligned}$$

3. An archaeologist discovers a clay pot full of ashes in an ancient tomb. The ashes contain a radioactive isotope of radium, Ra^{226}, which decays into an isotope of lead, Pb^{210}. This lead isotope is itself radioactive, and also decays.

(*i*) Let $R(t)$ and $L(t)$ respectively denote the amounts of radium Ra^{226} and lead Pb^{210} present in the ashes at a time t years after the fire in which they were formed. Explain briefly why $R(t)$ and $L(t)$ should satisfy the differential equations

(a) $\dfrac{dR}{dt} = -\lambda R$ (b) $\dfrac{dL}{dt} = \lambda R - \mu L$

where λ and μ are the decay constants of Ra^{226} and Pb^{210} respectively.

(*ii*) Let R_0 denote the amount of radium Ra^{226} present in the ashes initially (i.e., immediately after the fire in which they were formed) and suppose it is believed that the ashes initially contained no lead. Solve (*i*)(a) to obtain an expression for $R(t)$. Then solve (*i*)(b) to obtain an expression for $L(t)$.

(*iii*) The half-life of Ra^{226} is 1590 years, while that of Pb^{210} is 22 years. Use this information to determine the values of the decay constants λ and μ.

(*iv*) The ashes are found to contain 90 atoms of Ra^{226} for every atom of Pb^{210}. Deduce the age of the ashes (still assuming that they initially contained no Pb^{210}).

Solution

(*i*) (a) The rate at which Ra^{226} decays is proportional to the amount of Ra^{226} actually present; i.e., $-dR/dt$ is proportional to R. In fact the decay rate λ is defined to be the constant of proportionality, so $-dR/dt = \lambda R$ or

$$\frac{dR}{dt} = -\lambda R,$$

as required.

(b) The amount of Pb^{210} present increases at a rate dL/dt equal to the rate at which it is formed by the decay of Ra^{226} (λR as seen above), minus the rate at which it decays (which is μL, being proportional to the amount of Pb^{210} present). Hence

$$\frac{dL}{dt} = \lambda R - \mu L,$$

as required.

(*ii*) The solution of (a) is just $R(t) = R_0 e^{-\lambda t}$. Hence (b) can be written in the linear form,

$$\frac{dL}{dt} + \mu L = \lambda R_0 e^{-\lambda t}.$$

Multiplying both sides of this equation by the integrating factor $I(t) = e^{\mu t}$, we get

$$\frac{d}{dt}\big(e^{\mu t} L\big) = \lambda R_0 e^{(\mu-\lambda)t}.$$

Integration then gives

$$e^{\mu t} L = \frac{\lambda R_0}{\mu - \lambda} e^{(\mu-\lambda)t} + C,$$

and hence

$$L(t) = \frac{\lambda R_0}{\mu - \lambda} e^{-\lambda t} + C e^{-\mu t},$$

where C is an undetermined constant of integration. Taking $t = 0$ gives

$$L(0) = \frac{\lambda R_0}{\mu - \lambda} + C,$$

from which we deduce that $C = -\lambda R_0/(\mu - \lambda)$ (since we are told that $L(0) = 0$.) Thus the solutions for both R and L are

$$R(t) = R_0 e^{-\lambda t}, \qquad L(t) = \frac{\lambda R_0}{\mu - \lambda}\big(e^{-\lambda t} - e^{-\mu t}\big).$$

(*iii*) Recall that the decay constants can be obtained by dividing $\ln 2$ by the half-life; thus

$$\lambda = \frac{\ln 2}{1590} = 4.3594 \times 10^{-4}/\text{year}, \qquad \mu = \frac{\ln 2}{22} = 3.1507 \times 10^{-2}/\text{year}.$$

(*iv*) According to the results above, the number of Ra^{226} atoms present at time t for each Pb^{210} atom will be

$$\frac{R(t)}{L(t)} = \frac{\mu - \lambda}{\lambda} \frac{1}{1 - e^{(\lambda-\mu)t}}.$$

Thus if t_A is the age of the ashes, then

$$90 = \frac{\mu - \lambda}{\lambda} \frac{1}{1 - e^{(\lambda-\mu)t_A}},$$

and hence

$$1 - e^{(\lambda-\mu)t_A} = \frac{\mu - \lambda}{90\lambda} = 0.7919,$$

from which it follows that

$$t_A = -\frac{\ln(1 - 0.7919)}{\mu - \lambda} = \frac{1.5698}{0.03107} = 50.52 \text{ years}.$$

So the ashes are of relatively recent origin, which is to be expected on account of the relatively low abundance of lead.