THE UNIVERSITY OF SYDNEY
MATH1903 INTEGRAL CALCULUS AND MODELLING (ADVANCED)

Semester 2 **Tutorial Solutions Week 12** 2009

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1. Find the general solution of each of the following DEs:

(a) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 5y = 0.$ (b) $\dfrac{d^2x}{dt^2} - 6\dfrac{dx}{dt} + 9x = 0.$

(c) $\dfrac{d^2y}{dt^2} + 9y = 0.$ (d) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 25y = 0.$

Solution

(a) The auxiliary equation $m^2 + 4m - 5 = 0$ has roots $m = -5, 1$, and so the general solution is $y = Ae^{-5x} + Be^{x}$.

(b) The auxiliary equation $m^2 - 6m + 9 = 0$ has repeated roots $m = 3, 3$, and so the general solution is $x = Ae^{3t} + Bte^{3t}$.

(c) The auxiliary equation $m^2 + 9 = 0$ has complex roots $m = \pm 3i$, and so the general solution is $y = C\cos 3t + D\sin 3t$.

(d) The auxiliary equation $m^2 - 6m + 25 = 0$ has complex roots $m = 3 \pm 4i$, and so the general solution is $y = e^{3x}(C\cos 4x + D\sin 4x)$.

2. (a) Find the general solution of each of the following non-homogeneous differential equations:

(*i*) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x^2$ (*ii*) $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = 6e^{t}$ (*iii*) $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = 6e^{-t}$

(b) For the solutions of (i), (ii), (iii) find the particular solution satisfying the initial conditions

(*i*) $y(0) = y'(0) = 4$. (*ii*) $y(0) = 1$ and $\dot{y}(0) = 0$. (*iii*) $y(0) = 2$ and $\dot{y}(0) = 1$.

Solution

(a) (*i*) The auxiliary equation $m^2 - 2m + 1 = 0$ has a double root $m = 1$, and so the general solution of the complementary equation (also called the homogeneous equation) is $y_c = Ae^{x} + Bxe^{x}$. For a particular solution, try $y_p = ax^2 + bx + c$. Substituting this into the DE gives

$$2a - 2(2ax + b) + (ax^2 + bx + c) = x^2.$$

Comparing coefficients of like powers gives $a = 1$, $b - 4a = 0$ and $2a - 2b + c = 0$, and hence $a = 1$, $b = 4$ and $c = 6$. So a particular solution is $y_p = x^2 + 4x + 6$, and the general solution is

$$y = (A + Bx)e^{x} + x^2 + 4x + 6.$$

(*ii*) The auxiliary equation $m^2 + 3m + 2 = 0$ has roots $m = -1, -2$, and so the general solution of the complementary equation is $y_c = Ce^{-t} + De^{-2t}$. For a particular solution, try $y_p = \alpha e^{t}$. Substituting this into the DE gives $\alpha(e^{t} + 3e^{t} + 2e^{t}) = 6e^{t}$, which implies $\alpha = 1$. So a particular integral is $y_p = e^{t}$, and the general solution is

$$y = Ce^{-t} + De^{-2t} + e^{t}.$$

(*iii*) The auxiliary equation and hence the general solution of the complementary equation are the same as in the last part. In this case, however, the non-homogeneous term is itself a solution of the complementary equation and so we will not be able to produce a particular solution of the form αe^{-t}. The standard procedure in this case is to include a factor t. So a suitable trial solution will take the form $y_p = \alpha te^{-t}$. Substitution into the DE gives

$\alpha(t-2)e^{-t}+3\alpha(1-t)e^{-t}+2\alpha te^{-t} = 6e^{-t}$, which implies $\alpha = 6$. So a particular solution is $y_p = 6te^{-t}$, and the general solution is

$$y = (6t+C)e^{-t}+De^{-2t}.$$

(b) (*i*) The solution above gives $y(0) = A+6$ and $y'(0) = A+B+4$. So $y(0) = 4$ and $y'(0) = 4$ imply that $A = -2$ and $B = 2$, and so the required particular solution is $y = 2(x-1)e^x+x^2+4x+6$.

(*ii*) The solution above gives $y(0) = C+D+1$ and $\dot{y}(0) = -C-2D+1$. So $y(0) = 1$ and $\dot{y}(0) = 0$ imply that $C = -1$ and $D = 1$, and so the required particular solution is $y = -e^{-t}+e^{-2t}+e^t$.

(*iii*) The solution above gives $y(0) = C+D$ and $\dot{y}(0) = 6-C-2D$. So $y(0) = 2$ and $\dot{y}(0) = 1$ imply that $C = -1$ and $D = 3$, and so the required particular solution is $y = (6t-1)e^{-t}+3e^{-2t}$.

3. Show that the general solution of the 2nd order linear homogeneous differential equation with constant coefficients such that the auxiliary equation has complex roots $\lambda_1 = \bar{\lambda}_2 = \alpha+i\beta$ can be written in the from $y(x) = re^{\alpha x}\cos(\beta x+\phi)$ with arbitrary real constants r and ϕ.

Solution

The general (real) solution in this case is $y(x) = e^{\alpha x}(A\cos\beta x+B\sin\beta x)$. It is easiest to use complex notation. Then $y(x) = e^{\alpha x}\Re((A-iB)e^{i\beta x})$. Now write the complex constant $A-iB$ in polar form $re^{i\phi}$ and use the laws of the exponential function to find $y(x) = e^{\alpha x}\Re(re^{i(\beta x+\phi)}) = re^{\alpha x}\cos(\beta x+\phi)$. Replacing ϕ by $\tilde{\phi}-\pi/2$ we can obtain sine instead of cosine.

4. (a) Find the general solution of the non-homogeneous differential equation,

$$\frac{d^2y}{dt^2}+25y = 100\sin\omega t,$$

and the particular solution subject to the initial conditions $y(0) = 0$ and $\dot{y}(0) = 0$.

(b) For what positive value of ω does your solution break down? What physical phenomenon does this correspond to?

(c) Find the corresponding particular solution of the DE for this special value of ω by two methods:

(*i*) by the method of undetermined coefficients appropriate to this case (look for short cuts);

(*ii*) by fixing t in the result of part (a) and taking the limit as ω approaches its special value. You should meet a 0/0-type limit, which can be handled with l'Hôpital's rule.

Solution

(a) The auxiliary equation $m^2+25 = 0$ has roots $m = \pm 5i$, and so the general solution of the complementary equation is $y_c = C\cos 5t+D\sin 5t$. Since the non-homogeneous term is sinusoidal, we try a particular solution of the form, $y_p = \alpha\sin\omega t+\beta\cos\omega t$. This will work as long as $\omega \neq \pm 5$, which we assume for the present. Now, we can save ourselves some trouble by dropping the $\cos\omega t$ term in y_p. This is permitted because there is no first-order (or any odd-order) derivative term in the differential equation and because only a $\sin\omega t$ term appears on the right-hand side. (If you have any doubt about this, keep the cosine term in y_p and find that its coefficient is zero after a calculation.) Substituting $y_p = \alpha\sin\omega t$ into the DE gives $-\alpha\omega^2\sin\omega t+25\alpha\sin\omega t = 100\sin\omega t$, from which it follows that $\alpha = 100/(25-\omega^2)$. Thus, a particular solution is $y_p = 100(25-\omega^2)^{-1}\sin\omega t$, and the general solution is

$$y = C\cos 5t+D\sin 5t+\frac{100}{25-\omega^2}\sin\omega t.$$

We want the particular solution such that $y(0) = \dot{y}(0) = 0$. Differentiation of the general solution gives

$$\dot{y} \;=\; -\,5C\sin 5t + 5D\cos 5t \;+\; \frac{100\omega}{25-\omega^2}\cos\omega t.$$

The initial conditions imply that $C = 0$ and $D = -\,20\omega/(25-\omega^2)$. Hence the required particular solution is

$$y \;=\; \frac{100\sin\omega t - 20\omega\sin 5t}{25-\omega^2}\,.$$

(b) This solution breaks down when $\omega = \pm\,5$, the positive value being $\omega = 5$. The physical phenomenon is *resonance.* It can be understood as follows. The given differential equation is a typical differential equation governing a dynamical system having a natural frequency of vibration $5/(2\pi)$ which is subject to an external periodic force of frequency $\omega/(2\pi)$. When the two frequencies are different, the solution is a superposition of two periodic waves and stays bounded. When the frequencies are close, the amplitude is large. When the frequencies coincide, and in the absence of damping terms, the amplitude of the vibration grows until the system crashes in one way or another. Engineers need to be careful to design buildings, bridges, machines, etc., to avoid large amplitudes of vibration caused by resonance. On the other hand, when enough damping is present and forcing terms are controllable, resonance can be used to amplify a desired signal.

(c) *(i)* In the case $\omega = 5$, the previous trial solution $y_p = \alpha\sin\omega t + \beta\cos\omega t$ is no good because this is now a solution of the homogeneous equation. The standard trick in this case is to include a factor t, in which case $y_p = \alpha t\sin 5t + \beta t\cos 5t$. As before, we can simplify the problem by a symmetry argument. Because there is no first-order derivative in the differential equation and because the forcing term is an odd function, we can get away with restricting y_p to be an odd function. Thus $y_p = \beta t\cos 5t$. Its derivatives are $\dot{y}_p = \beta(-\,5t\sin 5t + \cos 5t)$ and $\ddot{y}_p = \beta(-\,25t\cos 5t - 10\sin 5t)$. Substituting into the differential equation and cancelling terms shows that $\beta = -10$. Hence a particular solution is $y_p = -\,10t\cos 5t$, and the general solution is

$$y \;=\; (C - 10t)\cos 5t + D\sin 5t.$$

Its derivative is $\dot{y} = (50t - 5C)\sin 5t + (5D - 10)\cos 5t$. The initial conditions are satisfied by $C = 0$ and $D = 2$. Hence the required particular solution is

$$y \;=\; 2\sin 5t - 10t\cos 5t.$$

(ii) If one puts $\omega = 5$ in the result of part (a), the solution becomes a $0/0$-type indeterminate form. L'Hôpital's rule can be used to take the limit $\omega \to 5$. Here, we must hold t constant while we take derivatives with respect to ω. Thus, in the case of resonance,

$$\begin{aligned} y &= \lim_{\omega\to 5}\frac{100\sin\omega t - 20\omega\sin 5t}{25-\omega^2} \\ &= \lim_{\omega\to 5}\frac{(\partial/\partial\omega)(100\sin\omega t - 20\omega\sin 5t)}{(\partial/\partial\omega)(25-\omega^2)} \\ &= \left.\frac{100t\cos\omega t - 20\sin 5t}{-\,2\omega}\right|_{\omega=5} \\ &= \frac{100t\cos 5t - 20\sin 5t}{-\,10} \\ &= 2\sin 5t - 10t\cos 5t. \end{aligned}$$

Of course, the two methods give the same answer. The factor $10t$ shows that the amplitude grows without bound.

5. (a) Find the general solution for each of the following higher-order linear equation (the prime denotes d/dx):

(i) $y''' - 3y' + 2y = 0$ *(ii)* $y''' = 8y$

(iii) $y'''' + 8y'' + 16y = 0$ *(iv)* $y''' - 3y' + 2y = 12\cosh x$

(b) Find a particular solution of the following differential equations:

(i) $y'' + y' + 2y = 8x^3$ *(ii)* $y'''' + 3y''' + 10y = 10e^{2x}$

Solution

(a) *(i)* $y''' - 3y' + 2y = 0$. The auxiliary equation is $m^3 - 3m + 2 = 0$. An obvious root is $m = 1$. Hence $m - 1$ is a factor. Then

$$m^3 - 3m + 2 = (m-1)(m^2 + m - 2) = (m-1)^2(m+2).$$

The roots are $m = 1, 1, -2$. Hence, the general solution of the DE is

$$y = (C_1 + C_2 x)e^x + C_3 e^{-2x}.$$

(ii) $y''' = 8y$. The auxiliary equation is $m^3 - 8 = (m-2)(m^2 + 2m + 4) = 0$. The roots are $m = 2$ and $m = -1 \pm i\sqrt{3}$. Hence, the general solution is

$$y = C_1 e^{2x} + e^{-x}\big(C_2\cos(\sqrt{3}\,x) + C_3\sin(\sqrt{3}\,x)\big).$$

(iii) $y'''' + 8y'' + 16y = 0$. The auxiliary equation is $m^4 + 8m^2 + 16 = (m^2 + 4)^2 = 0$. The roots are $m = 2i, 2i, -2i, -2i$. Hence, the general solution is

$$y = (C_1 + C_2 x)\cos 2x + (C_3 + C_4 x)\sin 2x.$$

(iv) The complementary function was found in part *(i)*. The right-hand side is $R(x) = 12\cosh x = 6e^x + 6e^{-x}$, which consists of two terms of the form Ae^{bx}. The case $b = 1$ is a double root of the auxiliary equation $(m-1)^2(m+2) = 0$. Hence, both e^x and xe^x appear in the complementary function. The part of the particular integral corresponding to the term $6e^x$ on the right-hand side will take the form Dx^2e^x. The case $b = -1$ is not a root of the auxiliary equation. Hence, the part of the particular integral corresponding to the term $6e^{-x}$ on the right-hand side will take the form Ee^{-x}. So, according to the method of undetermined coefficients,

$$\begin{aligned} y_p &= Dx^2e^x + Ee^{-x}, \\ y_p' &= D(x^2 + 2x)e^x - Ee^{-x}, \\ y_p'' &= D(x^2 + 4x + 2)e^x + Ee^{-x}, \\ y_p''' &= D(x^2 + 6x + 6)e^x - Ee^{-x}. \end{aligned}$$

Substituting into the differential equation gives

$$\begin{aligned} y_p''' - 3y_p' + 2y_p &= D(x^2 + 6x + 6 - 3x^2 - 6x + 2x^2)e^x + E(-1 + 3 + 2)e^{-x} \\ &= 6De^x + 4Ee^{-x}. \end{aligned}$$

This is to be identified with $6e^x + 6e^{-x}$. Hence $D = 1$ and $E = 3/2$. Thus a particular solution is $y_p = x^2e^x + (3/2)e^{-x}$. Adding on the complementary function gives the general solution:

$$y = (C_1 + C_2 x + x^2)e^x + C_3 e^{-2x} + \frac{3}{2}e^{-x}.$$

The last term can be replaced by $3\cosh x$ or $-3\sinh x$ (Why?).

(b) (*i*) $y''+y'+2y = 8x^3$. According to the method of undetermined coefficients, a particular cubic polynomial will satisfy this equation. So let

$$\begin{aligned} y_p &= Ax^3+Bx^2+Cx+D, \\ y_p' &= 3Ax^2+2Bx+C, \\ y_p'' &= 6Ax+2B, \\ y_p''+y_p'+2y_p &= 2Ax^3+(3A+2B)x^2+(6A+2B+2C)x+(2B+C+2D). \end{aligned}$$

The right-hand side of the last line is to be identified with $8x^3$. Thus,

$$2A=8, \qquad 3A+2B=0, \qquad 6A+2B+2C=0, \qquad 2B+C+2D=0.$$

Solving these in turn gives $A=4,\ B=-6,\ C=-6$ and $D=9$. So a particular solution is

$$y_p = 4x^3-6x^2-6x+9.$$

(*ii*) $y''''+3y'''+10y = 10e^{2x}$. Since 2 is not a root of the auxiliary equation $m^4+3m^3+10=0$, we may look for a particular solution of the form, $y_p = De^{2x}$. The left-hand side is $D(2^4+3\cdot 2^3+10)e^{2x} = 50De^{2x}$. Identifying this with $10e^{2x}$ gives $D=1/5$. Hence, a particular integral is

$$y_p = \frac{1}{5}e^{2x}.$$

In this example and others like it, we do not need to find the roots of the auxilairy equation.

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MATH1903 INTEGRAL CALCULUS AND MODELLING (ADVANCED)

Semester 2 **Further Exercises Solutions Week 12** 2009

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1. (a) Find the general solutions of the following second-order equations:

$$(i)\quad y'' + y' - 2y = x + 1 \qquad (ii)\quad y'' + 16y = e^x \qquad (iii)\quad y'' + y' - 2y = e^x$$

$$(iv)\quad y'' - 6y' + 9y = \sin x \qquad (v)\quad y'' - 6y' + 9y = e^{3x}.$$

(b) Find the particular solution of the differential equation $y'' - 6y' + 9y = e^{3x}$ which satisfies the initial conditions $y(0) = 1$ and $y'(0) = 0$.

Solution

(a) (i) The auxiliary equation is $m^2 + m - 2 = 0$, which has roots $m = 1, -2$. So the general solution of the homogeneous equation is $y_c = Ce^x + De^{-2x}$ for arbitrary constants C and D. To find a particular solution of the non-homogeneous equation, try $y_p = ax + b$. Then we need $x + 1 = y'' + y' - 2y = -2ax + (a - 2b)$. Hence $a = -1/2$ and $b = -3/4$, and so a suitable particular solution is $y_p = -(2x + 3)/4$. Hence, the general solution is

$$y = Ce^x + De^{-2x} - \frac{2x+3}{4}.$$

(ii) The auxiliary equation is $m^2 + 16 = 0$, which has roots $m = \pm 4i$. So the general solution of the homogeneous equation is $y_c = A\cos 4x + B\sin 4x$, for arbitrary constants A and B. To find a particular solution of the non-homogeneous equation, try $y_p = ae^x$. Then we need $e^x = y'' + 16y = ae^x + 16ae^x = 17ae^x$. So $a = 1/17$, and a particular solution is $y_p = e^x/17$. Hence, the general solution is

$$y = A\cos 4x + B\sin 4x + \frac{e^x}{17}.$$

(iii) As in part (i), the general solution of the homogeneous equation is $y_c = Ce^x + De^{-2x}$ for arbitrary constants C and D. To find a particular solution of the non-homogeneous equation, it is useless to try $y_p = ce^x$, because that is a solution of the homogeneous equation. So try instead $y_p = cxe^x$. Then we need $e^x = y'' + y' - 2y = 3ce^x$, and so $c = 1/3$. Hence a particular solution is $y_p = xe^x/3$, and the general solution is

$$y = Ce^x + De^{-2x} + \frac{xe^x}{3}.$$

(iv) The auxiliary equation is $m^2 - 6m + 9 = 0$, which has the repeated root $m = 3$. So the general solution of the homogeneous equation is $y_c = Ce^{3x} + Dxe^{3x}$, for arbitrary constants C and D. To find a particular solution of the non-homogeneous equation, try $y_p = a\cos x + b\sin x$. Then we need $\sin x = y'' - 6y' + 9y = (8a - 6b)\cos x + (6a + 8b)\sin x$. Hence we need $8a = 6b$ and $6a + 8b = 1$. Solving, we find that $a = 3/50$ and $b = 2/25$. Hence a particular solution is $y_p = (3\cos x + 4\sin x)/50$, and the general solution is

$$y = (C + Dx)e^{3x} + \frac{3\cos x + 4\sin x}{50}.$$

(v) As in part (iv), the general solution of the homogeneous equation is $y_c = Ce^{3x} + Dxe^{3x}$, for arbitrary constants C and D. To find a particular solution of the non-homogeneous equation, the possibility $y_p = (ax^2 + bx + c)e^{3x}$ suggests itself, but we can reduce this to $y_p = ax^2e^{3x}$ because $(bx + c)e^{3x}$ is a solution of the homogeneous equation. Routine calculations then show that $y'' - 6y' + 9y = 2ae^{3x}$, and so we must take $a = 1/2$. So a particular solution is

$y_p = x^2 e^{3x}/2$, and the general solution is

$$y = \Big\{C + Dx + \frac{x^2}{2}\Big\}e^{3x}.$$

(b) In part (a)(v), we have

$$y = \Big\{C + Dx + \frac{x^2}{2}\Big\}e^{3x},$$

$$y' = \Big\{3C + 3Dx + \frac{3x^2}{2} + D + x\Big\}e^{3x}.$$

Hence $y(0) = C$ and $y'(0) = 3C + D$. So the conditions $y(0) = 1$ and $y'(0) = 0$ imply that $C = 1$ and $D = -3$. Hence, the required particular solution is

$$y = \Big\{1 - 3x + \frac{x^2}{2}\Big\}e^{3x}.$$

2. An electrical circuit comprises a capacitor, a resistor, and an inductor connected in series to a voltage generator with a prescribed voltage $V(t)$. In terms of the electric charge $Q(t)$ and the current $I(t) = dQ/dt$, the voltage drops across the components are Q/C for a capacitance C, IR for a resistance R, and $L\,dI/dt$ for an inductance L. The total voltage drop is equal to that supplied by the generator,

$$\frac{Q}{C} + IR + L\frac{dI}{dt} = V(t).$$

(a) Using $I = dQ/dt$, derive the equation,

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \frac{dV}{dt}.$$

(b) Find the general solution of the complementary equation, distinguishing three possible types of behaviour, depending on the values of $R^2 - 4L/C$. Show that, provided $L, R > 0$, all solutions die out as $t \to \infty$.

(c) Find a particular solution for an AC voltage source $V(t) = V_0\cos(\omega_0 t)$.

Solution

(a) Just differentiate the given equation $(Q/C) + IR + L\,dI/dt = V(t)$, using the tacit hypothesis that C, R and L are constants, and the fact that $dQ/dt = I$, and we get the desired equation.

(b) The auxiliary equation is $Lm^2 + Rm + 1/C = 0$, whose roots are

$$m_1, m_2 = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}.$$

Case 1: $R^2 > 4L/C$. Then $0 < R^2 - 4L/C < R^2$ (assuming $L, C > 0$), and so $\sqrt{R^2 - 4L/C} < R$. Hence the numerator of both m_1 and m_2 is negative. Thus $m_1, m_2 < 0$. Hence the general solution of the complementary equation

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = 0$$

is

$$I(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t},$$

which clearly tends to 0 as $t \to \infty$, because $m_1, m_2 < 0$.
Case 2: $R^2 < 4L/C$. Then the two roots m_1 and m_2 are

$$m_1, m_2 = -\frac{R}{2L} \pm i\omega, \quad \text{where} \quad \omega = \frac{\sqrt{4L/C - R^2}}{2L}.$$

Now the general solution of the complementary equation is

$$I(t) = e^{-Rt/(2L)}\big(c_1\cos\omega t + c_2\sin\omega t\big).$$

This clearly tends to 0 as $t \to \infty$, because $-R/(2L) < 0$.

Case 3: $R^2 = 4L/C$. Then the only root of the auxiliary equation is $-R/(2L)$, this being a double root. Now the general solution of the complementary equation is

$$I(t) \;=\; e^{-Rt/(2L)}\bigl(c_1 + c_2 t\bigr).$$

This clearly tends to 0 as $t \to \infty$, because $-R/(2L) < 0$. We use here the fact that $xe^{-x} \to 0$ as $x \to \infty$, as you can see from L'Hopital's rule, for example.

(c) According to the method of undetermined coefficients, there will be a particular solution of the form, $I(t) = A\cos\omega_0 t + B\sin\omega_0 t$ for some A and B to be determined. Take this $I(t)$ and calculate $L\,d^2I/dt^2 + R\,dI/dt + I/C$, and set this equal to $V_0\cos\omega_0 t$. The result is

$$\bigl\{-L\omega_0^2 A + R\omega_0 B + A/C\bigr\}\cos\omega_0 t \;+\; \bigl\{-L\omega_0^2 B - R\omega_0 A + B/C\bigr\}\sin\omega_0 t \;=\; V_0\cos\omega_0 t.$$

Equating the coefficients of $\cos\omega_0 t$ and $\sin\omega_0 t$ and solving for A and B gives

$$A \;=\; \frac{C(1 - CL\omega_0^2)V_0}{(1 - CL\omega_0^2)^2 + (CR\omega_0)^2}\,, \qquad B \;=\; \frac{C^2 R\omega_0 V_0}{(1 - CL\omega_0^2)^2 + (CR\omega_0)^2}\,.$$

Hence the particular solution of the postulated form is

$$I(t) \;=\; CV_0\,\frac{(1 - CL\omega_0^2)\cos\omega_0 t + CR\omega_0\sin\omega_0 t}{(1 - CL\omega_0^2)^2 + (CR\omega_0)^2}\,.$$

Adding the complementary solution and giving the integration constants particular values yields an infinite supply of particular solutions. All of these differ from the solution just found by exponentially decaying terms when $R > 0$ and $L > 0$. Physically, this means that, when an AC circuit is switched on, there is, in general, a brief transient current after which the current settles into a steady sinusoidal form with the same frequency as the applied voltage.

3. (a) Show that both $I_1(x) = e^x$ and $I_2(x) = e^{3x}$ are integrating factors for the differential equation,

$$y'' + 4y' + 3y \;=\; 4\,\frac{1 - x + x^2}{(1 + x^2)^2}\,e^{-x}.$$

(Note that the method of undetermined coefficients is not appropriate for this equation, so don't attempt to use it. This type of DE will not appear on the exam paper.)

(b) Use the integrating factor $I_1(x)$ to obtain a first integral of the equation in part (a). This first integral will take the form of a linear differential equation of the first order for y in which a constant of integration appears in the coefficients. Then solve the first-order DE to get the general solution of the DE in part (a). (You may get an integral that does not seem elementary—arrange the integrand as the derivative of a product.)

(c) Do the same with the integrating factor $I_2(x)$. Give different names to the constants of integration, e.g., C_3 and C_4.

(d) Obtain y by eliminating y' from the first integrals in parts (b) and (c). This is another way to get the general solution.

(e) Here is a fourth method, known as variation of parameters. Observe that the complementary equation is solved by $y_c = Ae^{-x} + Be^{-3x}$. Attempt a trial solution of the full equation in part (a) of the form $y = A(x)e^{-x} + B(x)e^{-3x}$ and let $A'(x)e^{-x} + B'(x)e^{-3x} = 0$. Substitute into the DE in part (a) and cancel terms to get another constraint involving $A'(x)$ and $B'(x)$. Solve for $A'(x)$ and $B'(x)$ and integrate to get $A(x)$ and $B(x)$, picking up two constants of integration along the way. Deduce the general solution for y. Make sure all versions of the general solution agree (except possibly for the names of the integration constants).

Solution

(a) The complementary equation $y'' + 4y' + 3y = 0$ has the auxiliary equation $m^2 + 4m + 3 = 0$, whose roots are $m = -1, -3$. It follows that $y_1 = e^{-x}$ and $y_2 = e^{-3x}$ are particular solutions of the complementary equation and their reciprocals $I_1(x) = e^x$ and $I_2(x) = e^{3x}$ are integrating factors for the full equation.

Remark. More generally, for any nth-order linear DE, $y^{(n)} + \cdots = R(x)$, if m is any root of the auxiliary equation, then the DE can be written $(d/dx - m)\{y^{(n-1)} + \cdots\} = R(x)$. Any DE in this form, whether linear or nonlinear, will have an integrating factor e^{-mx} because multiplication by e^{-mx} yields a DE, $(d/dx)\{e^{-mx}(y^{(n-1)} + \cdots)\} = e^{-mx}R(x)$, which can be directly integrated at least once.

(b) Let $R(x)$ denote the right-hand side of the given equation. To take advantage of the integrating factor $I_1(x) = e^x$, write the DE as

$$\left(\frac{d}{dx} + 1\right)\{y' + 3y\} \;=\; R(x).$$

Multiplying both sides by e^x puts the DE in the form,

$$\frac{d}{dx}\{e^x(y' + 3y)\} \;=\; e^x R(x),$$

which is now directly integrable. Substituting for $R(x)$ and integrating gives

$$\begin{aligned} e^x(y' + 3y) &= \int 4\,\frac{1 - x + x^2}{(1 + x^2)^2}\,dx \\ &= 4\tan^{-1} x + \frac{2}{1 + x^2} + C_1\,, \end{aligned}$$

where C_1 is a constant of integration. We have therefore obtained a first integral,

$$\frac{dy}{dx} + 3y \;=\; e^{-x}\left\{4\tan^{-1} x + \frac{2}{1 + x^2} + C_1\right\}.$$

This is a linear DE of the first order with integrating factor e^{3x}. Multiplying both sides by e^{3x} puts the DE in the form,

$$\frac{d}{dx}\{e^{3x}y\} \;=\; e^{2x}\left\{4\tan^{-1} x + \frac{2}{1 + x^2} + C_1\right\}.$$

The first two terms on the right-hand side form the derivative of a product. Hence the next integration can be accomplished, giving

$$e^{3x}y \;=\; 2e^{2x}\tan^{-1} x + \frac{1}{2}C_1 e^{2x} + C_2\,,$$

where C_2 is a second constant of integration. Hence, the general solution of the DE in part (a) is

$$y \;=\; 2e^{-x}\tan^{-1} x + \frac{1}{2}C_1 e^{-x} + C_2 e^{-3x}.$$

(Ordinarily, we would rename C_1 to get rid of the factor 1/2 but we will leave it untouched because we want to compare different derivations of the general solution.)

(c) We proceed as in part (b). To take advantage of the integrating factor $I_2(x) = e^{3x}$, write the DE as

$$\left(\frac{d}{dx} + 3\right)\{y' + y\} \;=\; R(x).$$

Multiplying both sides by e^{3x} puts the DE in the form,

$$\frac{d}{dx}\{e^{3x}(y' + y)\} \;=\; e^{3x}R(x),$$

which is now directly integrable. Substituting for $R(x)$ and integrating gives

$$\begin{aligned} e^{3x}(y' + y) &= \int 4e^{2x}\,\frac{1 - x + x^2}{(1 + x^2)^2}\,dx \\ &= \frac{2e^{2x}}{1 + x^2} + C_3\,, \end{aligned}$$

where C_3 is a constant of integration. We have therefore obtained another first integral,

$$\frac{dy}{dx} + y \;=\; \frac{2e^{-x}}{1 + x^2} + C_3 e^{-3x}.$$

This linear DE has integrating factor e^x and can therefore be written in the form,

$$\frac{d}{dx}\{e^x y\} = \frac{2}{1+x^2} + C_3 e^{-2x}.$$

The integration is easy, yielding the general solution,

$$y = 2e^{-x}\tan^{-1}x - \frac{1}{2}C_3 e^{-3x} + C_4 e^{-x}.$$

This agrees with the solution found in part (b) if we identify $C_3 = -2C_2$ and $C_4 = (1/2)C_1$.

(d) Collecting the first integrals found in the previous two parts, we have

$$y' + 3y = e^{-x}\Big\{4\tan^{-1}x + \frac{2}{1+x^2} + C_1\Big\},$$

$$y' + y = \frac{2e^{-x}}{1+x^2} + C_3 e^{-3x}.$$

Eliminating y' by subtraction, and dividing the result by 2, we get

$$y = 2e^{-x}\tan^{-1}x + \frac{1}{2}C_1 e^{-x} - \frac{1}{2}C_3 e^{-3x}.$$

This is the general solution of the DE in part (a) again. It agrees with the solutions found in parts (b) and (c) with the same identifications of constants.

(e) According to the method of variation of parameters, if the complementary equation is solved by $y_c = Ay_1 + By_2$, then the full non-homogeneous equation can be solved by postulating a particular solution of the form $y = A(x)y_1 + B(x)y_2$, where we have replaced the constants of integration by undetermined functions. In the case of the DE in part (a), the method runs as follows. Start with

$$y = A(x)e^{-x} + B(x)e^{-3x}$$

and take a derivative:

$$y' = A'(x)e^{-x} + B'(x)e^{-3x} - A(x)e^{-x} - 3B(x)e^{-3x}.$$

Since we have two undetermined functions and only one DE to constrain them, we can set one additional constraint according to convenience. That constraint is

$$A'(x)e^{-x} + B'(x)e^{-3x} = 0.$$

Then the last equation simplifies to

$$y' = -A(x)e^{-x} - 3B(x)e^{-3x}.$$

Taking another derivative gives

$$y'' = -A'(x)e^{-x} - 3B'(x)e^{-3x} + A(x)e^{-x} + 9B(x)e^{-3x}.$$

Substituting y, y' and y'' into the DE and cancelling terms gives the second constraint:

$$-A'(x)e^{-x} - 3B'(x)e^{-3x} = R(x).$$

The two constraints are just algebraic equations for $A'(x)$ and $B'(x)$. Solving then gives

$$A'(x) = \frac{1}{2}e^{x}R(x), \qquad B'(x) = -\frac{1}{2}e^{3x}R(x).$$

The integrals are cases that we have already met in parts (b) and (c). The results are

$$A(x) = 2\tan^{-1}x + \frac{1}{1+x^2} + C_5\,, \qquad B(x) = -\frac{e^{2x}}{1+x^2} + C_6\,,$$

where C_5 and C_6 are constants of integration. (Omitting the constants would make y a particular solution; leaving them in makes y the general solution.) Thence, the general solution is

$$y = A(x)e^{-x} + B(x)e^{-3x} = 2e^{-x}\tan^{-1}x + C_5 e^{-x} + C_6 e^{-3x}.$$

This agrees with the previous versions of the general solution if we identify $C_5 = (1/2)C_1 = C_4$ and $C_6 = C_2 = -(1/2)C_3$.

Remark. There are more methods available which the interested student might like to pursue. For example, the changes of variable $y = e^{-x}v$ and $y = e^{-3x}w$.