THE UNIVERSITY OF SYDNEY

MATH1903 INTEGRAL CALCULUS AND MODELLING (ADVANCED)

Semester 2 **Tutorial Solutions Week 13** 2009

1. Find the general solution of the pair of differential equations,

$$\frac{dx}{dt} = 5x - 3y, \qquad \frac{dy}{dt} = 2x,$$

by differentiating the first equation and then using the second to obtain a linear second-order DE for $x(t)$. (Your solution should have *two* arbitrary constants of integration.) Find the particular solution satisfying the initial conditions $x = 2$, $y = 1$ when $t = 0$.

Solution

Differentiating the first equation gives

$$\frac{d^2x}{dt^2} = 5\frac{dx}{dt} - 3\frac{dy}{dt} = 5\frac{dx}{dt} - 6x,$$

where we have used the fact that $dy/dt = 2x$. Hence $x(t)$ satisfies $\ddot{x} - 5\dot{x} + 6x = 0$. The auxiliary equation is $m^2 - 5m + 6 = 0$, which has roots $m = 3, 2$, and so the general solution for x is

$$x = Ae^{3t} + Be^{2t}.$$

Hence $\dot{x} = 3Ae^{3t} + 2Be^{2t}$. But the first-order equation $\dot{x} = 5x - 3y$ implies that

$$\begin{aligned} y &= \frac{1}{3}\big(5x - \dot{x}\big) = \frac{1}{3}\big(5Ae^{3t} + 5Be^{2t} - 3Ae^{3t} - 2Be^{2t}\big) \\ &= \frac{2}{3}Ae^{3t} + Be^{2t}. \end{aligned}$$

At time $t = 0$ we therefore have $x = A + B$ and $y = (2/3)A + B$. But we are told that $x = 2$ and $y = 1$ when $t = 0$, which implies that $A = 3$ and $B = -1$. So the particular solution is

$$x(t) = 3e^{3t} - e^{2t}, \qquad y = 2e^{3t} - e^{2t}.$$

2. The DE of the previous problem can be written in matrix-vector form as

$$\frac{d}{dt}z = Mz, \quad z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad M = \begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix}.$$

The general solution is given by $z(t) = e^{Mt}z_0$. Compute e^{Mt} explicitly in this example following these steps:

(a) Find the eigenvalues of M, i.e. the roots of the characteristic polynomial $\det(M - \lambda I)$.

(b) Find the eigenvectors of M, i.e. the vectors v_i that satisfy $Mv_i = \lambda_i v_i$.

(c) Form the matrix T whose columns are v_i and verify that $T^{-1}MT = D$ where D is diagonal with eigenvalues λ_1, λ_2 as diagonal entries.

(d) Finally compute $e^{Mt} = Te^{Dt}T^{-1}$ where $e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$.

(e) Verify that for the initial condition $z_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ the solution $e^{Mt} z_0$ is the same as found in the previous problem.

Solution

(a) $\det(M - \lambda I) = (5 - \lambda)(-\lambda) + 6 = \lambda^2 - 5\lambda + 6 = 0$ gives $\lambda_{1,2} = 2, 3$.

(b) $Mv = 2v$ has solution $v_1 = (3, 2)^t$ and $Mv = 3v$ has solution $v_2 = (1, 1)^t$. Eigenvectors are only defined up to multiplication by a non-zero number, but the given choice is the one where v_i have the smallest positive integer entries.

(c) The matrix T is $\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$, and $T^{-1} = \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}$ such that in deed $T^{-1}MT = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. Note that if you put the eigenvectors in a different order in T then the eigenvalues in D come out in the corresponding different order.

(d) Multiplying the three matrices together gives $e^{Mt} = \begin{pmatrix} -2e^{2t} + 3e^{3t} & 3e^{2t} - 3e^{3t} \\ -2e^{2t} + 2e^{3t} & 3e^{2t} - 2e^{3t} \end{pmatrix}$

(e) Multiplying the matrix e^{Mt} from the previous part by $(2, 1)^t$ in deed gives the particular solution $(x(t), y(t))^t$ found in problem one.

3. The following is a model for an arms race between two superpowers X and Y. Denote the level of preparation of X for war by $x(t)$ and that of Y by $y(t)$, where t represents time. The model consists of the equations,

$$\frac{dx}{dt} = -ax + by, \qquad \frac{dy}{dt} = cx - dy,$$

where a, b, c and d are positive constants.

(a) Eliminate y to obtain a second-order homogeneous linear DE for $x(t)$.

(b) Write down the auxiliary equation of this DE, and its general solution in terms of its roots m_1 and m_2. Show that if $ad > bc$ then both roots of the auxiliary equation must be negative. What does this suggest about the likelihood of war?

(c) Suppose $a = d = 1$, $b = c = 3$, and that $x = 5$, $y = 1$ at the initial time $t = 0$. Find the particular solution for x and y. What can you conclude about the likelihood of war in this case?

(d) Briefly discuss the assumptions underlying the model. Do you think these assumptions are realistic?

Solution

(a) Differentiate the first equation with respect to time to obtain

$$\frac{d^2x}{dt^2} = -a\frac{dx}{dt} + b\frac{dy}{dt},$$

substitute dy/dt from the second,

$$\frac{d^2x}{dt^2} = -a\frac{dx}{dt} + b(cx - dy),$$

and eliminate by by using the first equation again, to get

$$\frac{d^2x}{dt^2} = -a\frac{dx}{dt} + bcx - d\left(\frac{dx}{dt} + ax\right).$$

Rewriting this DE in the standard form, we have

$$\frac{d^2x}{dt^2} + (a+d)\frac{dx}{dt} + (ad-bc)x = 0.$$

(b) The auxiliary equation is

$$m^2 + (a+d)m + (ad-bc) = 0,$$

with roots

$$m_{1,2} = \frac{-(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} = \frac{-(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}.$$

The second of these expressions makes it clear that both roots are real, so the general solution is

$$x = Ae^{m_1 t} + Be^{m_2 t}.$$

From the auxiliary equation we see that the sum of the roots $m_1 + m_2 = -(a+d)$ is negative, while the product of the roots $m_1 m_2 = ad - bc$ is positive, if $ad > bc$. Thus both roots must be negative.

The solution therefore predicts that X and Y prepare less and less for war. This suggests that war would be increasingly unlikely.

(c) With $a = d = 1$, $b = c = 3$, the roots of the auxiliary equation are $2, -4$. Thus the general solution for $x(t)$ is

$$x = Ae^{-4t} + Be^{2t},$$

and so $\dot{x} = -4Ae^{-4t} + 2Be^{2t}$. But the first of the two original equations now reads $\dot{x} = -x + 3y$, and hence

$$y = \frac{1}{3}\left(\frac{dx}{dt} + x\right) = \frac{1}{3}\left(-4Ae^{-4t} + 2Be^{2t} + Ae^{-4t} + Be^{2t}\right)$$
$$= -Ae^{-4t} + Be^{2t}.$$

At time $t = 0$ we therefore have $x = A + B$ and $y = -A + B$. But we are told that $x = 5$ and $y = 1$ when $t = 0$, which implies that $A = 2$ and $B = 3$. So the particular solution is

$$x(t) = 2e^{-4t} + 3e^{2t}, \qquad y = -2e^{-4t} + 3e^{2t}.$$

The exponential growth of the terms involving e^{2t} ensures that both X and Y become ever more prepared for war, and so the danger of war may well increase. (However, without a more detailed model, it is not clear that increased military spending will result in a war; it may instead lead to the economic collapse of one of the superpowers.)

(d) The model assumes that there is no limit on the rate at which military spending can grow. In reality, an exponentially growing military budget cannot be sustained for very long, as the superpowers only have limited resources. The model will inevitably break down as the level of expenditure becomes unaffordable.

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MATH1903 INTEGRAL CALCULUS AND MODELLING (ADVANCED)

Semester 2	**Further Exercises Solutions Week 13**	2009

1. Consider the system of linear first order equations

$$\dot{z} = Mz$$

where z is a vector and M a square matrix of the same dimension. The general solution is given by $z(t) = e^{Mt}z_0$. In order to compute e^{Mt} we need to compute powers of M. When M can be diagonalised by a matrix T then we have $M^i = TD^iT^{-1}$ where $D = T^{-1}MT$ is diagonal. This problem is about the case when M cannot be diagonalised. For a 2×2 non-diagonalisable matrix M there exist an invertible matrix T such that

$$T^{-1}MT = J \quad \text{where} \quad J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

(a) Derive the formula $M^i = TJ^iT^{-1}$ for powers of M.

(b) Proof by induction that $J^i = \begin{pmatrix} \lambda^i & i\lambda^{i-1} \\ 0 & \lambda^i \end{pmatrix}$.

(c) Use these two results to sum the series of e^{Mt} in this case. Thus show that

$$e^{Mt} = T \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} T^{-1}.$$

(d) Re-derive the result using the following observation: $M = \lambda I + N$ where $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Because I and N commute, $IN = NI$, we can write $e^{(\lambda I+N)t} = e^{\lambda It}e^{Nt}$. Now observe that N is nilpotent, i.e. N^2 is the zero-matrix, and hence e^{Nt} is very simple.

(e) When solved via a 2nd order equation, which case does this correspond to?

Solution

(a) We are given that $M = TJT^{-1}$, and hence $M^i = TJT^{-1}TJT^{-1} \dots TJT^{-1} = TJJ \dots JT^{-1} = TJ^iT^{-1}$.

(b) The formula is obviously true for $i = 1$. The induction step is

$$J^{i+1} = J^iJ = \begin{pmatrix} \lambda^i & i\lambda^{i-1} \\ 0 & \lambda^i \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^{i+1} & \lambda^i + i\lambda^{i-1}\lambda \\ 0 & \lambda^{i+1} \end{pmatrix} = \begin{pmatrix} \lambda^{i+1} & (i+1)\lambda^i \\ 0 & \lambda^{i+1} \end{pmatrix}.$$

(c) As in the diagonalisable case the matrices T and T^{-1} can be extracted from the sum. The essentially new part is the sum $e^{Jt} = \sum_{i=0}^{\infty} J^it^i/i!$. In the diagonal we obtain $e^{\lambda t}$ as before. The sum of the non-zero off-diagonal terms is

$$\sum_{i=0}^{\infty} i\lambda^{i-1}t^i/i! = t\sum_{i=1} \lambda^{i-1}t^{i-1}/(i-1)! = t\sum_{i=0}^{\infty} \lambda^it^i/i! = te^{\lambda t}$$

(d) From the information given we have $e^{Jt} = e^{\lambda It}e^{Nt}$. Now $e^{Nt} = \sum (Nt)^i/i!$ but $N^2 = 0$ and hence $N^i = 0$ for $i \geq 2$. Therefore $e^{Nt} = I + Nt$. Also note that $e^{\lambda It} = Ie^{\lambda t}$. Hence $e^{Jt} = e^{\lambda t}I(I + Nt) = e^{\lambda t}(I + Nt)$, which is the desired result.

(e) The eigenvalues of M are the same as the eigenvalues of J (because they are related by a similarity transformation). Since J is upper triangular, the eigenvalues of J are the diagonal entries, hence λ and λ again. So this is the case where the auxiliary equation has a double root, so it is case (iii). Alternatively observe that e^{Jt} contains exponentials multiplied by t, which again only appears in case (iii).

2. Consider the pair of differential equations

$$\frac{dx}{dt} = 7x - 2y, \qquad \frac{dy}{dt} = 2x + 3y.$$

(a) Obtain a second-order differential equation for $x(t)$ and find its general solution.

(b) Find the associated general solution for $y(t)$.

(c) Show that, if $x(0) > y(0)$, then $x(t)$ and $y(t)$ increase without limit as $t \to \infty$. Conversely, if $x(0) < y(0)$, show that $x(t)$ and $y(t)$ decrease without limit as $t \to \infty$.

(d) Find the particular solution for $x(t)$ and $y(t)$ satisfying the initial conditions $x(0) = 2$, $\dot{y}(0) = 1$.

(e) Confirm explicitly that the expressions you have obtained for $x(t)$ and $y(t)$ obey the first order equations given in part (a).

Solution

(a) Differentiating the first equation gives $\ddot{x} = 7\dot{x} - 2\dot{y}$. Then using the second equation to eliminate $\dot{y}$ gives

$$\ddot{x} = 7\dot{x} - 2(2x + 3y) = 7\dot{x} - 4x - 6y.$$

But the first equation implies that $y = (7/2)x - (1/2)\dot{x}$, and so the expression for $\ddot{x}$ becomes

$$\ddot{x} = 7\dot{x} - 4x - 3(7x - \dot{x}) = 10\dot{x} - 25x,$$

which is written conveniently in the form

$$\ddot{x} - 10\dot{x} + 25x = 0.$$

The auxiliary equation $m^2 - 10m + 25 = 0$ has a double root $m = 5$, and so the general solution for $x(t)$ is

$$x(t) = Ae^{5t} + Bte^{5t}.$$

(b) The first DE implies that $y = (7/2)x - (1/2)\dot{x}$. Hence, with the general solution for $x(t)$ given above, we find that

$$y(t) = \frac{7}{2}\left(Ae^{5t} + Bte^{5t}\right) - \frac{1}{2}\left(5Ae^{5t} + 5Bte^{5t} + Be^{5t}\right) = \left(A - \frac{B}{2}\right)e^{5t} + Bte^{5t}.$$

(c) The solutions above give $x(0) = A$ and $y(0) = A - (1/2)B$, and hence $x(0) - y(0) = (1/2)B$. If $x(0) > y(0)$, then $B > 0$, and so we see from the expressions given above that $x(t)$ and $y(t)$ will increase without limit as $t \to \infty$. On the other hand, if $x(0) < y(0)$, then $B < 0$, and we see from the expressions given above that $x(t)$ and $y(t)$ will decrease without limit as $t \to \infty$. (Note that the value of A is immaterial, since for any choice of A we find that $|Bte^{5t}| > |Ae^{5t}|$ and $|Bte^{5t}| > |(A - B/2)e^{5t}|$ for sufficiently large values of t.)

(d) The expressions above for $x(t)$ and $y(t)$ give $x(0) = A$ and $\dot{y}(0) = 5A - (3/2)B$. Hence the initial conditions $x(0) = 2$, $\dot{y}(0) = 1$ indicate that we must set $A = 2$ and $B = 6$. Thus the particular solutions are

$$x(t) = 2e^{5t} + 6te^{5t} = (6t + 2)e^{5t}, \qquad y(t) = -e^{5t} + 6te^{5t} = (6t - 1)e^{5t}.$$

(e) Differentiating, we get

$$\dot{x} = (30t + 16)e^{5t}, \qquad \dot{y}(t) = (30t + 1)e^{5t}.$$

Moreover,

$$7x - 2y = (42t + 14)e^{5t} - (12t - 2)e^{5t} = (30t + 16)e^{5t}$$

and

$$2x + 3y = (12t + 4)e^{5t} + (18t - 3)e^{5t} = (30t + 1)e^{5t}.$$

Thus $\dot{x} = 7x - 2y$ and $\dot{y} = 2x + 3y$, as required.

3. For the system of two first order equation $\dot{z} = Mz$ with general matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, find a criterion on the matrix M such that the origin is an asymptotically stable equilibrium.

Solution

The eigenvalues are given by the roots of $\lambda^2 - \operatorname{tr} M\lambda + \det M = 0$. They are $2\lambda_\pm = \operatorname{tr} M \pm \sqrt{(\operatorname{tr} M)^2 - 4\det M}$. If the eigenvalues of M are complex ($\operatorname{tr} M^2 - 4\det M < 0$), the real part of λ is given by $\operatorname{tr} M/2 = (a + d)/2$. If we have a repeated eigenvalue ($(\operatorname{tr} M)^2 - 4\det M = 0$), the eigenvalues is given by $\lambda = \operatorname{tr} M/2$. Hence in both cases we need to require $\operatorname{tr} M < 0$.

When $\operatorname{tr} M^2 - 4\det M > 0$ the eigenvalues are real, so they both need to be less than zero. The condition $\operatorname{tr} M < 0$ assures that $\lambda_- < 0$. To assure $\lambda_+ < 0$ in addition we need $\det M < 0$.