Semester 2 **Tutorial Solutions Week 8** 2009

1. Find the general solution for each of the following differential equations:

$(i)\quad \dfrac{dy}{dx} = 1 + \sin x + \sin^2 x, \qquad (ii)\quad x^3\dfrac{dy}{dx} = 2x^2 + 5, \quad x > 0,$

$(iii)\quad \dfrac{dy}{dx} = \dfrac{1}{\cosh y}, \qquad (iv)\quad \dfrac{dy}{dx} = \cos y, \quad |y| < \dfrac{\pi}{2}.$

In the last case, obtain y explicitly (several equivalent expressions).

Solution

(i) $y = \displaystyle\int \left(1 + \sin x + \sin^2 x\right) dx = \tfrac{3}{2}x - \cos x - \tfrac{1}{4}\sin 2x + C,$

or, equivalently, $y = \dfrac{3}{2}x - \cos x - \dfrac{1}{2}\sin x \cos x + C.$

(ii) $\dfrac{dy}{dx} = 2x^{-1} + 5x^{-3}, \ x > 0,$

and so

$$y = \int \left(2x^{-1} + 5x^{-3}\right) dx = 2\ln x - \frac{5}{2x^2} + C.$$

(iii) $\dfrac{dx}{dy} = \cosh y$

and so

$$x = \int \cosh y \, dy = \sinh y + C.$$

Hence, the general solution is $y = \sinh^{-1}(x - C)$.

(iv) $\dfrac{dx}{dy} = \sec y, \ -\pi/2 < y < \pi/2,$

and so

$$x = \int \sec y \, dy = \ln(\sec y + \tan y) + C.$$

Inverting the logarithm and renaming the constant gives

$$\sec y + \tan y = e^{x-C} = Ae^x, \qquad A > 0.$$

Multiplying both sides by $\cos y$ and squaring gives

$$(1 + \sin y)^2 = A^2 e^{2x}(1 - \sin^2 y).$$

Cancelling the common factor $1 + \sin y$ and solving for $\sin y$ gives

$$\sin y = \frac{A^2 e^{2x} - 1}{A^2 e^{2x} + 1}, \qquad \cos y = \frac{2Ae^x}{A^2 e^{2x} + 1},$$

where we also deduced the value of $\cos y$ from the earlier equation for $\sec y + \tan y$. The expressions on the right can also be expressed in terms of hyperbolic functions by reinstating the earlier integration constant C according to $A = e^{-C}$. We have now arrived at four equivalent expressions for the general solution of the differential equation:

$$y = \sin^{-1}\left(\frac{A^2e^{2x}-1}{A^2e^{2x}+1}\right), \qquad y = \sin^{-1}\{\tanh(x-C)\},$$

$$y = \cos^{-1}\left(\frac{2Ae^{x}}{A^2e^{2x}+1}\right), \qquad y = \cos^{-1}\{\operatorname{sech}(x-C)\}.$$

Note that the first pair is preferable because they are valid for all real x and show that y is an odd function of $x - C$. The second pair is only valid for $x \geq C$ because the inverse cosine gives the wrong quadrant for $x < C$. (Nevertheless, if this was an exam or assignment question, you would get full marks for any one of these four expressions.)

2. Find the particular solutions of the following differential equations satisfying the given conditions:

(i) $\dfrac{dy}{dx} = 1 - 2x - 3x^2, \quad y = -1$ when $x = 1$.

(ii) $e^{2x}\dfrac{dy}{dx} + 1 = 0, \quad y \to 2$ as $x \to \infty$.

(iii) $\dfrac{dy}{dx} = \dfrac{y}{2} + \dfrac{1}{2y}, \quad y = 2$ when $x = 0$.

Solution

(i) $y = \displaystyle\int (1 - 2x - 3x^2)\, dx = x - x^2 - x^3 + C.$

With $x = 1$ we get $y = 1 - 1^2 - 1^3 + C = C - 1$, and so the condition implies $C = 0$. Hence the particular solution is

$$y = x - x^2 - x^3.$$

Since this is an initial value problem we can also use definite integrals

$$y = -1 + \int_1^x (1 - 2\bar{x} - 3\bar{x}^2)d\bar{x} = -1 + [\bar{x} - \bar{x}^2 - \bar{x}^3]_1^x = x - x^2 - x^3$$

(ii) $\dfrac{dy}{dx} = -e^{-2x}$ and so $y = \displaystyle\int \left(-e^{-2x}\right) dx = \frac{1}{2}e^{-2x} + C.$

We see that $y \to C$ as $x \to \infty$, so the condition gives $C = 2$. The particular solution is therefore

$$y = \frac{1}{2}e^{-2x} + 2.$$

(iii) $\dfrac{dy}{dx} = \dfrac{y^2+1}{2y}$ so $\dfrac{dx}{dy} = \dfrac{2y}{y^2+1}$ and $x = \displaystyle\int \frac{2y\ dy}{y^2+1} = \ln(y^2+1) + C.$

When $y = 2$ we get $x = \ln 5 + C$, and so the condition gives $C = -\ln 5$. Thus the particular solution is

$$x = \ln(y^2+1) - \ln 5 = \ln\{(y^2+1)/5\}.$$

Solving for y (and choosing the positive square root so that $y(0) = 2$) gives

$$y = \sqrt{5e^x - 1}\,.$$

3. (i) According to Newton's law mass times acceleration equals force. Consider a particle with mass m and velocity v in the field of gravity which exerts a force $-mg$. Using that acceleration is dv/dt write down the differential equation for v and solve it with initial condition $v(0) = 0$.

(*ii*) Use the solution of the first part and the fact that $dx/dt = v$ to find the position $x(t)$ when initially $x(0) = h$.

(*iii*) Repeat the two previous parts assuming that in addition to the force of gravity there is also a friction force that is proportional to the velocity and opposing the velocity. This is a good model for small velocities. Find the termial speed $v_\infty = |\lim_{t\to\infty} v(t)|$ and use v_∞ to eliminate the proportionality constant of the friction term from the solution.

(*iv*) Compute the Taylor polynomials $T_3(t)$ of $x(t)$ for the two cases. Hence verify that for small times the solutions with and without friction are close to each other.

Solution

(*i*) Integrating $mdv/dt = -mg$ with respect to time gives $v(t) = -gt + C$ and from $v(0) = 0$ we find $C = 0$. Note that negative velocity means the motion is downward.

(*ii*) Integrating $dx/dt = v = -gt$ with respect to time gives $x(t) = -gt^2/2 + C$ and from $x(0) = h$ we find $C = h$.

(*iii*) The DE with friction is $mdv/dt = -mg - cv$. Note that since $v < 0$ the friction does decrease the magnitude of the force, as it should. The signs are arranged that that the positive direction is up. This DE cannot simply be integrated with respect to time as before. Instead consider the DE for the inverse function $t(v)$

$$\frac{dt}{dv} = \frac{-1}{c/mv + g}$$

and integration gives

$$-t = \frac{m}{c}\log(c/mv + g) + C\,.$$

Solving for $v(t)$ and renaming $e^C = 1/A$ gives

$$v(t) = -mg/c + A\exp(-c/mt)\,.$$

The initial condition $v(0) = 0$ gives $0 = -mg/c + A$ so that $v(t) = -mg/c(1 - \exp(-c/mt))$. The large t limit is $v_\infty = |\lim_{t\to\infty} v(t)| = mg/c$, and eliminating c gives

$$v(t) = -v_\infty(1 - \exp(-gt/v_\infty))\,.$$

Now the DE for position $dx/dt = v$ can be solved by integrating. Using definite integrals gives $x(t) - x(0) = \int_0^t v(t)dt$ and hence

$$\begin{aligned} x(t) &= h - v_\infty \left[t + v_\infty/g\exp(-gt/v_\infty)\right]_0^t \\ &= h - v_\infty(t + v_\infty/g(\exp(-gt/v_\infty) - 1)) \end{aligned}$$

(*iv*) $T_3(t)$ for the solution without friction is $x(t) = h - gt^2/2$ (nothing to do), while with friction we find $x' = v$, $x'' = v' = -g - c/mv$, $x''' = v'' = -c/mv' = -c/m(-g - c/mv)$. The computation of the derivatives is simplified by using the differential equation. Now use $x(0) = h$ and $v(0) = 0$ so that $x'(0) = 0$, $x''(0) = -g$, $x'''(0) = cg/m$ and finally

$$T_3(x) = h - g\frac{t^2}{2} + \frac{cg}{m}\frac{t^3}{3!}\,.$$

Hence the difference is cubic in t. This shows that with friction (and for small t) heavier bodies fall faster.

1. An upright cylindrical tank of radius R has a small hole of area a at a height x above the base. The speed at which water flows out is given by Torricelli's law as $v = \sqrt{2g(h-x)}$, where h is the height of the water surface above the base. The rate at which the volume of the water in the tank decreases is therefore $va = a\sqrt{2g(h-x)}$.

(a) Use this to show that h satisfies the differential equation,

$$\frac{dh}{dt} = -k\sqrt{h-x},$$

where $k = a\sqrt{2g}/(\pi R^2)$.

(b) Solve to find $h(t)$, given that the water level is initially at $h = H$. How long does it take for the water surface to fall to the level of the hole? How long does it take when the hole is at the base of the tank?

Solution

(a) The volume V of water in the tank changes at the rate

$$\frac{dV}{dt} = -a\sqrt{2g(h-x)},$$

the negative sign indicating that the volume is decreasing. (Note that x is a fixed constant in this exercise.) To obtain a differential equation for the height h of the water level, we note that the volume of water in the cylindrical tank is $V = \pi R^2 h$. Hence,

$$\frac{dV}{dt} = \frac{d}{dt}\left(\pi R^2 h\right) = \pi R^2 \frac{dh}{dt}.$$

Comparing this with the earlier equation we get

$$\pi R^2 \frac{dh}{dt} = -a\sqrt{2g(h-x)} \quad \text{or} \quad \frac{dh}{dt} = -k\sqrt{h-x},$$

where $k = a\sqrt{2g}/(\pi R^2)$.

(b) Separating and integrating the equation, we get

$$\int \frac{dh}{\sqrt{h-x}} = -k\int dt \Rightarrow 2\sqrt{h-x} = -kt + C.$$

Since $h = H$ when $t = 0$, we must have $2\sqrt{H-x} = C$. Hence,

$$t = \frac{2\left(\sqrt{H-x} - \sqrt{h(t)-x}\right)}{k} \quad \text{or} \quad h(t) = x + \left\{\sqrt{H-x} - \tfrac{1}{2}kt\right\}^2.$$

The water level therefore reaches the hole at $h = x$ after a time,

$$t = \frac{2\sqrt{H-x}}{k} = \frac{2\pi R^2}{a}\sqrt{\frac{H-x}{2g}}.$$

(Remember to return to the original parameters given in the question.) When the hole is at the base of the tank ($x = 0$), this time is $(2\pi R^2/a)\sqrt{H/(2g)}$.

2. According to the Gompertz model, the population N of a colony of animals grows according to the differential equation,

$$\frac{dN}{dt} = \beta N \ln\left(\frac{M}{N}\right),$$

where M is the maximum sustainable population size and β is a positive constant.

(i) Solve this differential equation. (ii) Find $\lim_{t\to\infty} N(t)$.

(iii) Find the particular solution for which $N = M/2$ when $t = 0$.

Solution

(i) We can rewrite the equation as

$$\begin{aligned}\beta\frac{dt}{dN} &= \frac{1/N}{\ln(M/N)}\\ &= -\frac{1/N}{\ln(N/M)} = -\frac{(d/dN)\ln(N/M)}{\ln(N/M)} = -\frac{d}{dN}\ln|\ln(N/M)|,\end{aligned}$$

and then integration with respect to N gives

$$\beta t = -\ln|\ln(N/M)| + C,$$

where C is an undetermined constant. We rearrange this to give $\ln|\ln(N/M)| = C - \beta t$, and hence

$$\ln(N/M) = \pm e^C e^{-\beta t} = Ae^{-\beta t},$$

where $A = \pm e^C$. It follows that

$$N(t) = M\exp\left(Ae^{-\beta t}\right).$$

(ii) As $t \to \infty$, we have $Ae^{-\beta t} \to 0$; hence $\exp\left(Ae^{-\beta t}\right) \to 1$ and $N(t) \to M$.

(iii) Taking $t = 0$ gives $N = M\exp\left(Ae^0\right) = e^A M$. But we are told that $N(0) = M/2$ and so it follows that $A = -\ln 2$. Thus, the particular solution is

$$N(t) = M\exp\left(-\ln 2\, e^{-\beta t}\right) = M/2^{(e^{-\beta t})}.$$

3. Denote the solution without fricition in Q3(i) by $v_n(t)$ and the solution with friction in Q3(iii) by $v_f(t)$. Show that $|v_n(t)| > |v_f(t)|$ for $t > 0$.

Solution

Since both velocities are negative we need to show that

$$gt > v_\infty(1 - \exp(-gt/v_\infty))$$

for $t > 0$. Introduce $x = gt/v_\infty$ as a new variable, then the expression becomes

$$v_\infty(x - 1 + \exp(-x)) > 0\,.$$

Now observe that $1 - x$ is the Taylor polynomial $T_1(x)$ of $\exp(-x)$, so that

$$\exp(-x) - T_1(x) = R_1(x)$$

and all we need to show is that $R_1(x) > 0$ for $x > 0$. But

$$R_1(x) = \frac{1}{2}\exp(-c)x^2 \quad \text{where } 0 \le c \le x$$

is clearly positive, which concludes the proof.

4. This exercise provides more practice in evaluating l'Hôpital-type limits using power series methods. Evaluate

$$(i)\quad \lim_{x\to 0}\frac{e^{-2x^2}-\cos 2x}{x^4}\,, \qquad (ii)\quad \lim_{x\to 0}\left\{\coth^2 x-\cot^2 x\right\},$$

$$(iii)\quad \lim_{x\to 0}\frac{\ln(1+2x)-2x\sqrt{1-2x}}{x-\sin x}\,.$$

Solution

(i) The denominator x^4 tells us that we need to calculate the power series expansions of the functions e^{-2x^2} and $\cos 2x$ in the numerator up to the x^4 terms, and we expect all the earlier terms to cancel for otherwise the limit would not exist. From the standard series $e^x = 1 + x + x^2/2! + \cdots$ and $\cos x = 1 - x^2/2! + x^4/4! - \cdots$, we get

$$e^{-2x^2} \;=\; 1-2x^2+2x^4-\cdots\,, \qquad \cos 2x \;=\; 1-2x^2+\tfrac{2}{3}x^4-\cdots\,.$$

Hence, we do indeed get the expected cancellation and

$$\lim_{x\to 0}\frac{e^{-2x^2}-\cos 2x}{x^4} \;=\; \lim_{x\to 0}\frac{(4/3)x^4+\cdots}{x^4} \;=\; \frac{4}{3}\,.$$

This example is not difficult to do with l'Hôpital's rule (four applications needed). We leave this as an exercise.

(ii) This is a limit of the type $(\infty-\infty)$. To sufficient accuracy, long division of power series gives

$$\coth x \;=\; \frac{\cosh x}{\sinh x} \;=\; \frac{1+x^2/2!+\cdots}{x+x^3/3!+\cdots} \;=\; \frac{1}{x}\left\{1+\frac{1}{3}x^2+\cdots\right\},$$

$$\cot x \;=\; \frac{\cos x}{\sin x} \;=\; \frac{1-x^2/2!+\cdots}{x-x^3/3!+\cdots} \;=\; \frac{1}{x}\left\{1-\frac{1}{3}x^2+\cdots\right\}.$$

Hence,

$$\coth^2 x \;=\; \frac{1}{x^2}\left\{1+\frac{2}{3}x^2+\cdots\right\}, \qquad \cot^2 x \;=\; \frac{1}{x^2}\left\{1-\frac{2}{3}x^2+\cdots\right\}.$$

The negative powers cancel and we get

$$\lim_{x\to 0}\left\{\coth^2 x-\cot^2 x\right\} \;=\; \frac{4}{3}\,.$$

Remark. To evaluate this limit with l'Hôpital's rule, you would need to rewrite the limit as a ratio of $0/0$ type:

$$\lim_{x\to 0}\left\{\coth^2 x-\cot^2 x\right\} \;=\; \lim_{x\to 0}\frac{\cosh^2 x\sin^2 x-\sinh^2 x\cos^2 x}{\sinh^2 x\sin^2 x}\,.$$

Four applications of l'Hôpital's rule are needed. A shortcut becomes apparent after two applications. Nevertheless, you will appreciate that the power series method is considerably faster.

(iii) The denominator is $x-\left\{x-x^3/3!+\cdots\right\}$ and so begins with the term $x^3/6$. It follows that we should expand the functions appearing in the numerator out to the x^3 terms. The logarithm and binomial series give

$$\ln(1+2x) \;=\; 2x-\tfrac{1}{2}(2x)^2+\tfrac{1}{3}(2x)^3-\cdots \;=\; 2x-2x^2+\tfrac{8}{3}x^3-\cdots\,,$$

$$\sqrt{1-2x} \;=\; 1+\tfrac{1}{2}(-2x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(-2x)^2+\cdots \;=\; 1-x-\tfrac{1}{2}x^2-\cdots\,.$$

Hence,

$$\begin{aligned}
&\ln(1+2x) - 2x\sqrt{1-2x} \\
&\qquad\qquad = \left\{2x - 2x^2 + \tfrac{8}{3}x^3 - \cdots\right\} - \left\{2x - 2x^2 - x^3 - \cdots\right\} \\
&\qquad\qquad = \tfrac{11}{3}x^3 + \cdots \ .
\end{aligned}$$

The required limit is

$$\lim_{x\to 0} \frac{\ln(1+2x) - 2x\sqrt{1-2x}}{x - \sin x} \ = \ \lim_{x\to 0} \frac{(11/3)x^3 + \cdots}{(1/6)x^3 + \cdots} \ = \ 22.$$

This problem could also be solved by three applications of l'Hôpital's rule with a bit of effort.