THE UNIVERSITY OF SYDNEY

MATH1903 INTEGRAL CALCULUS AND MODELLING (ADVANCED)

Semester 2 **Tutorial Solutions Week 9** 2009

1. Find the general solutions of

(a) $(1+x^2)\dfrac{dy}{dx} + xy = 0$, (b) $x\dfrac{dy}{dx} = y^2 - 1$,

(c) $(x^2y^2 + x^2 + y^2 + 1)\dfrac{dy}{dx} = xy + x$, (d) $ye^x\dfrac{dy}{dx} = y^2 + y - 2$.

Solution

(a)

$$\frac{dy}{dx} = -\left(\frac{x}{1+x^2}\right)y$$

$$\Rightarrow \int \frac{dy}{y} = -\int \frac{x\ dx}{1+x^2} \qquad (\text{“}+C\text{” built in})$$

$$\Rightarrow \ln|y| = -\frac{1}{2}\ln(1+x^2) + C.$$

With a new constant of integration $A = \pm e^C$, we get the general solution,

$$y = A\big(1+x^2\big)^{-1/2}.$$

(b)

$$\frac{dy}{dx} = \frac{y^2-1}{x}$$

$$\Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y^2-1} = \frac{1}{2}\int\left(\frac{1}{y-1} - \frac{1}{y+1}\right)dy$$

$$\Rightarrow \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = \ln|x| + C.$$

With a new constant $A = \pm e^{2C}$, exponentiation gives

$$\frac{y-1}{y+1} = Ax^2,$$

which can be rearranged to give the general solution,

$$y = \frac{1+Ax^2}{1-Ax^2}.$$

(c)

$$(x^2+1)(y^2+1)\frac{dy}{dx} = x(y+1)$$

$$\Rightarrow \int \frac{x\;dx}{x^2+1} = \int \frac{y^2+1}{y+1}\,dy = \int \left(y - 1 + \frac{2}{y+1}\right)\,dy$$

$$\Rightarrow \frac{1}{2}\ln\big(x^2+1\big) = \frac{1}{2}y^2 - y + 2\ln|y+1| + C.$$

Exponentiating and renaming the constant, we get the general solution,

$$(1+y)^4\,e^{y^2-2y} = A\big(1+x^2\big).$$

This relation defines $y(x)$ implicitly. There is no explicit expression for $y(x)$. (If desired, we could solve for x in terms of y.)

(d)

$$\frac{dy}{dx} = \frac{(y+2)(y-1)}{y}\,e^{-x}$$

$$\Rightarrow \int e^{-x}\,dx = \int \frac{y\;dy}{(y-1)(y+2)} = \frac{1}{3}\int \left[\frac{1}{y-1} + \frac{2}{y+2}\right]\,dy$$

$$\Rightarrow -e^{-x} + C = \frac{1}{3}\,\ln\big|(y-1)(y+2)^2\big|.$$

Exponentiating and renaming the constant, we get the general solution,

$$(y-1)(y+2)^2 = A\,\exp\big\{-3e^{-x}\big\}.$$

In this case, there is an explicit expression for $y(x)$, but it is messy. Leave the answer as a cubic equation for y as shown.

2. Find the particular solutions of

(a) $\dfrac{dy}{dx} = xe^{y-x^2}, \quad y(0)=0,$ (b) $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}, \quad y(0)=C.$

[The expression for the solution of (b) can be simplified using the formula $\tan(A+B) = (\tan A + \tan B)/(1 - \tan A \tan B)$, or, equivalently, the identity for inverse tangents from the assignment.]

Solution

(a)

$$\int e^{-y}dy = \int x\,e^{-x^2}\,dx$$

$$\Rightarrow -e^{-y} = -\frac{1}{2}e^{-x^2} + C_1$$

$$\Rightarrow y = -\ln\left(\frac{1}{2}\,e^{-x^2} + C\right),$$

where $C = -C_1$. Substituting $x = 0$ and $y = 0$ into this equation gives $0 = -\ln\left(\frac{1}{2}+C\right)$ and hence $C = 1/2$. So the particular solution is

$$y = -\ln\left\{\tfrac{1}{2}\left(e^{-x^2}+1\right)\right\}.$$

(b)

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

$$\Rightarrow \tan^{-1} y = \tan^{-1} x + C_1 .$$

Substituting $x = 0$ and $y = C$ into this equation gives $C_1 = \tan^{-1} C$. Hence,

$$\begin{aligned} y &= \tan\left(\tan^{-1} x + \tan^{-1} C\right) \\ &= \frac{\tan(\tan^{-1} x) + \tan(\tan^{-1} C)}{1 - \tan(\tan^{-1} x)\tan(\tan^{-1} C)} \\ &= \frac{C+x}{1-Cx} . \end{aligned}$$

[*Remark.* This differential equation has an interesting feature. Both the equation and its solution involve only rational functions while the method of obtaining the solution took a detour through the realm of transcendental functions. A similar example involving algebraic functions (left as an exercise) is $dy/dx = \sqrt{ay^2+b}/\sqrt{ax^2+b}$, whose general solution is $y = x\sqrt{1+aC^2} + C\sqrt{ax^2+b}$. It is remarkable that a similar construction works when the quadratic function under the square root is replaced by a cubic or quartic function, even though the intermediate transcendental functions are not elementary.]

3. Einstein's Theory of Relativity predicts the existence of black holes: regions in space from which nothing can escape, due to strong gravitational forces. The theory predicts that black holes will be formed when large stars collapse.

However, Einstein's theory did not take into account quantum mechanical effects. In 1975, Stephen Hawking used quantum theory to show that a black hole should glow slightly; that is, it should radiate energy and particles in the same way that a heated object does. Assuming that nothing else falls into the black hole, this causes its mass M to decrease at the rate governed by the differential equation,

$$\frac{dM}{dt} = -\frac{\alpha}{M^2} ,$$

where t denotes time and α is a constant whose value is not yet known precisely.

(a) Find the general solution $M(t)$ of this differential equation.

(b) Find the particular solution which satisfies the condition that the mass is M_0 when $t = 0$.

(c) How long does it take for a black hole which initially has mass M_0 to lose half its mass? How long does it take for it to evaporate completely?

Solution

(a) Rewriting the equation as $\alpha\, dt/dM = -M^2$ and integrating with respect to M gives $\alpha t = -\frac{1}{3}M^3 + C$, with C an arbitrary constant of integration. Thus

$$M(t) = (3C - 3\alpha t)^{1/3}.$$

(b) With $t = 0$, the solution above gives $M_0 = (3C)^{1/3}$. Hence $C = {M_0}^3/3$ and so the particular solution is

$$M(t) = \left({M_0}^3 - 3\alpha t\right)^{1/3}.$$

(c) We want to find $t_{1/2}$ such that $M(t_{1/2}) = \frac{1}{2}M_0$. Thus $\left({M_0}^3 - 3\alpha t_{1/2}\right)^{1/3} = \frac{1}{2}M_0$, which gives ${M_0}^3 - 3\alpha t_{1/2} = \frac{1}{8}{M_0}^3$ and hence $3\alpha t_{1/2} = \frac{7}{8}{M_0}^3$. Thus the time taken for the black hole to evaporate half its mass is

$$t_{1/2} = \frac{7{M_0}^3}{24\alpha}.$$

We also want to find t_1 such that $M(t_1) = 0$. Thus $\left({M_0}^3 - 3\alpha t_1\right)^{1/3} = 0$, which gives $3\alpha t_1 = {M_0}^3$. Hence the time taken to evaporate completely is

$$t_1 = \frac{{M_0}^3}{3\alpha}.$$

The rate of evaporation increases towards the end. In particular, $t_1 = (8/7)t_{1/2}$.

4. Let y be the number of people in a stable economy who have an income of x or more. The economist Vilfredo Pareto (1848–1923) discovered that the rate at which y decreases with increasing x is directly proportional to the number of people with income x or more and inversely proportional to the income x. Obtain a differential equation for $y(x)$ and solve it to find y in terms of x, given that the minimum income is x_0 and the total population is N.

Solution

According to Pareto, the rate of change of y with x is directly proportional to y and inversely proportional to x, i.e.,

$$\frac{dy}{dx} \propto \frac{y}{x} \quad \text{or} \quad \frac{dy}{dx} = -k\,\frac{y}{x}.$$

We have put in the negative sign because we are told that y is a decreasing function of x; the constant of proportionality k is then positive. Separating and integrating the differential equation gives

$$\int \frac{1}{y}\,dy = -k\int \frac{dx}{x} \Rightarrow \ln y = -k\ln x + C \Rightarrow y = Ax^{-k}.$$

Given that the minimum income is x_0, the whole population must have at least this income; i.e., $y = N$ when $x = x_0$, so that $N = Ax_0^{-k}$ or $A = N(x_0)^k$. Substituting for A gives the desired result, $y = N(x/x_0)^{-k}$.

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MATH1903 INTEGRAL CALCULUS AND MODELLING (ADVANCED)

Semester 2 **Further Exercises Solutions Week 9** 2009

1. Find the general solutions of

(a) $\dfrac{dy}{dx} = \dfrac{x+\sin x}{3y^2}$, (b) $\dfrac{dx}{dt} = 1+t-x-tx$, (c) $\dfrac{dy}{dx} = \dfrac{\ln x}{xy+xy^3}$.

Solution

(a) $\displaystyle\int 3y^2\,dy = \int (x+\sin x)\,dx \Rightarrow y^3 = \frac{1}{2}x^2 - \cos x + C \Rightarrow y = \Big(\frac{1}{2}x^2 - \cos x + C\Big)^{1/3}$.

(b) The right-hand side factorises as $(1+t)(1-x)$. Hence, the differential equation is separable. Separating and integrating gives

$$\int \frac{dx}{1-x} = \int (1+t)\,dt \Rightarrow -\ln|x-1| = t + \frac{1}{2}t^2 + C \Rightarrow x = Ae^{-t-t^2/2} + 1.$$

(c) $\displaystyle\int (y+y^3)\,dy = \int \frac{\ln x}{x}\,dx = \int u\,du$, where $u = \ln x$. Integrating gives $\frac{1}{2}y^2 + \frac{1}{4}y^4 = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$. Completing the square gives $(y^2+1)^2 = 2(\ln x)^2 + D$, where $D = 4C+1$. It follows that $y^2 + 1 = \pm\sqrt{2(\ln x)^2 + D}$, and so (ignoring the imaginary solution that results if we use the negative root),

$$y = \pm\sqrt{\sqrt{2(\ln x)^2 + D} - 1}\,.$$

2. (a) Find particular solutions satisfying the given conditions for:

(*i*) $\dfrac{dy}{dx} = \dfrac{1+x}{xy}$ $(x>0)$, $y(1) = -4$; (*ii*) $\dfrac{dy}{dt} = \dfrac{ty+3t}{t^2+1}$, $y(2) = 2$.

(b) Find a function $g(x)$ such that $g'(x) = g(x)\big(1+g(x)\big)$ and $g(0) = 1$.

(c) Find an equation of the curve that passes through the point $(1,1)$ and whose slope at (x,y) is y^2/x^3.

Solution

(a) (*i*) $\displaystyle\int y\,dy = \int\Big(\frac{1}{x}+1\Big)\,dx \Rightarrow \frac{1}{2}y^2 = \ln|x| + x + C$, and so the general solution is $y(x) = \pm\sqrt{2x + 2\ln|x| + 2C}$. We then get $y(1) = \pm\sqrt{2+2C}$, and so the initial condition $y(1) = -4$ implies that we must choose the negative sign and $C = 7$. So the required particular solution is $y(x) = -\sqrt{2x + 2\ln x + 14}$, valid for $x > 0$.

(*ii*) $\displaystyle\int \frac{1}{y+3}\,dy = \int \frac{t}{t^2+1}\,dt \Rightarrow \ln|y+3| = \frac{1}{2}\ln(t^2+1) + C$, and so the general solution is $y(t) = A\sqrt{t^2+1} - 3$. We then get $y(2) = A\sqrt{5} - 3$, and so the initial condition $y(2) = 2$ implies that we must choose $A = \sqrt{5}$. So the particular solution is $y(t) = \sqrt{5(t^2+1)} - 3$.

(b) The equation $dg/dx = g(1+g)$ is separable; separation and integration gives

$$\int \frac{dg}{g(1+g)} = \int dx,$$

which gives $\ln\left|g/(g+1)\right| = x + C$. Hence $(g+1)/g = Ae^{-x}$ and

$$g(x) = \frac{1}{Ae^{-x}-1}.$$

This is the general solution. The initial condition $g(0) = 1$ implies that $A = 2$. So the required particular solution is $g(x) = 1/(2e^{-x}-1)$.

(c) Separating and integrating the equation $dy/dx = y^2/x^3$ gives $\displaystyle\int \frac{dy}{y^2} = \int \frac{dx}{x^3}$ and hence, $\displaystyle -\frac{1}{y} = -\frac{1}{2x^2} + C$. So the general solution is $y(x) = 2x^2/(1-2Cx^2)$. We then get $y(1) = 2/(1-2C)$, and so the initial condition $y(1) = 1$ implies that $C = -1/2$. So the required particular solution is $y(x) = 2x^2/(1+x^2)$.

3. A molecule of substance A can combine with a molecule of substance B to form a molecule of substance X, in a reaction which is denoted $A + B \to X$. According to the Law of Mass Action, the rate of formation of X is proportional to the product of the amounts of A and B present. A test-tube initially contains amounts a and b of substances A and B, respectively, (measured in billions of molecules), but none of substance X.

(a) Let $x(t)$ denote the amount of substance X (i.e., the number of billions of X molecules) produced within the first t seconds. Write down a differential equation for $x(t)$.

(b) Assuming that $a \neq b$, solve this equation to obtain an expression for $x(t)$.

(c) Suppose that initially there are two molecules of B for every molecule of A, and that after 10 seconds there are six molecules of B for every molecule of A. What is the ratio after 30 seconds?

(d) The experiment is repeated, but with the initial amount of substance B halved so as to equal the initial amount a of substance A. (As before, substance X is absent initially.) What fraction of A molecules remain after 30 seconds?

Solution

(a) The amounts of substances A and B left after time t are, respectively, $a - x(t)$ and $b - x(t)$. We are told that the rate of increase of x is proportional to the product of these quantities, and so

$$\frac{dx}{dt} = k(a-x)(b-x),$$

where k is a positive constant which characterises the reaction $A + B \to X$.

(b) Separating, we get

$$\int \frac{dx}{(a-x)(b-x)} = \int k\,dt \quad \text{and hence} \quad \frac{1}{b-a}\ln\left|\frac{b-x}{a-x}\right| = kt + C,$$

where C is an undetermined constant of integration. Putting $A = \pm e^{(b-a)C}$ we then have

$$\frac{b-x(t)}{a-x(t)} = A\,e^{(b-a)kt},$$

and, since $x(0) = 0$, we deduce that $A = b/a$. Hence,

$$\frac{b-x}{a-x} = \frac{b}{a}\,e^{(b-a)kt}. \tag{1}$$

After rearrangement we see that

$$x(t) = \frac{ab\big(e^{(b-a)kt} - 1\big)}{be^{(b-a)kt} - a}.$$

(c) Equation (1) tells us how many molecules of B there are for each molecule of A after t seconds. But we know that $b = 2a$ (since there is initially twice as much of B as of A), and so this expression reduces to

$$\frac{2a-x}{a-x} = \frac{\text{number of molecules of } B \text{ after } t \text{ seconds}}{\text{number of molecules of } A \text{ after } t \text{ seconds}} = 2\,e^{akt}.$$

We are also told that this ratio is equal to 6 when $t = 10$, so $6 = 2e^{10ak}$, and hence $e^{10ak} = 3$. Thus, after 30 seconds, the number of molecules of B per molecule of A must be

$$2\,e^{30ak} = 2\big[e^{10ak}\big]^3 = 2 \times 3^3 = 54.$$

(d) The amount of substance A present initially is again a, and this time the amount of B present initially is also a. Thus, after t seconds, the test-tube will contain amounts $a - x(t)$ of both substances. Hence, the rate of formation of X is given by the differential equation,

$$\frac{dx}{dt} = k(a-x)^2,$$

where the constant k has the same value as in the previous experiment, where we found that $e^{10ak} = 3$ and hence $k = (\ln 3)/(10a)$. Separating and integrating this equation, we get

$$\frac{1}{a - x(t)} = kt + C_2\,,$$

where C_2 is a new constant of integration. Recalling that $x(0) = 0$, we deduce that $C_2 = 1/a$, and so the amount of A remaining at time t is

$$a - x(t) = \frac{a}{1 + akt} = \frac{a}{1 + (\ln 3)t/10}.$$

Dividing by a (the amount of A present initially), and taking $t = 30$, we see that the proportion of the original A molecules which remain after 30 seconds is

$$\frac{a - x(t)}{a} = \frac{1}{1 + (\ln 3) \times 30/10} = \frac{1}{1 + 3\ln 3} = \frac{1}{4.2958} = 0.23278\cdots.$$

Thus, about 23% of the A molecules present initially remain after 30 seconds.

4. [From 1998 exam] According to one model, the growth of a rabbit population on an uninhabited island is described by the differential equation,

$$\frac{dN}{dt} = \alpha\big(1 - \beta\cos 2\pi t\big)N(M - N),$$

where $N(t)$ denotes the population of rabbits after t years, and the constants M, α and β depend upon the size and location of the island.

(a) Explain briefly why you think the $\beta\cos 2\pi t$ term is included in this equation.

(b) Find the general solution of the equation given above.

(c) Show that, according to the model, the ratio $N/(M - N)$ should grow by a factor of $e^{\alpha M}$ over any one-year period.

(d) On a certain island the maximum sustainable population of rabbits is estimated to be 2700. The observed population was 450 on 8th March 1997 and reached 900 exactly one year later. According to the model, what will the population be on 8th March 1999?

Solution

(a) This term is presumably included to account for seasonal factors; its effect is to lower the predicted population growth rate at certain times of the year (presumably the winter).

(b) Separating and integrating we get

$$\int \frac{1}{M}\left(\frac{1}{N} - \frac{1}{N-M}\right) dN = \int \alpha\big(1 - \beta\cos 2\pi t\big)\, dt,$$

or

$$\frac{1}{M}\ln\left|\frac{N}{M-N}\right| = \alpha\Big(t - \frac{\beta}{2\pi}\sin 2\pi t\Big) + C,$$

from which it follows that

$$\frac{M-N}{N} = A\exp\left\{-\alpha M\Big(t - \frac{\beta}{2\pi}\sin(2\pi t)\Big)\right\},$$

where $A = \pm\, e^{-MC}$ is an undetermined constant. Hence,

$$N(t) = \frac{M}{1 + A\exp\big\{-\alpha M\big(t - (\beta/(2\pi))\sin(2\pi t)\big)\big\}}.$$

(c) From the expression obtained above for $(M - N)/N$, we see that

$$\frac{N(t)}{M - N(t)} = \frac{1}{A}\exp\left\{\alpha M\Big(t - \frac{\beta}{2\pi}\sin(2\pi t)\Big)\right\},$$

and hence,

$$\begin{aligned}\frac{N(t+1)}{M - N(t+1)} &= \frac{1}{A}\exp\left\{\alpha M\Big(t + 1 - \frac{\beta}{2\pi}\sin(2\pi(t+1))\Big)\right\}\\ &= \frac{1}{A}e^{\alpha M}\exp\left\{\alpha M\Big(t - \frac{\beta}{2\pi}\sin(2\pi t)\Big)\right\}\\ &= e^{\alpha M}\frac{N(t)}{M - N(t)},\end{aligned}$$

as required.

(d) With $M = 2700$ we see that $N/(M - N) = 450/(2700 - 450) = 1/5$ in 1997, while $N/(M - N) = 900/(2700 - 900) = 1/2$ in 1998. Thus, $N/(M - N)$ is increasing by a constant factor of 5/2 each year, and will therefore reach 5/4 by 8th March 1999. Solving for N on this date, we see that

$$\frac{M-N}{N} = \frac{4}{5} \Rightarrow N = \frac{5M}{9} = \frac{5}{9}\times 2700 = 1500.$$

Thus the rabbit population on 8th March 1999 will be 1500.