1. Which of the following differential equations are linear? Can any of the nonlinear cases be transformed into a linear differential equation by a simple change of variables? (You are not required to solve these DEs; however, all have elementary solutions and you should attempt them at home after the tutorial.)

(a) $\dfrac{dy}{dx} + \dfrac{3y}{x} = \sin x$

(b) $(x-1)^3\dfrac{dy}{dx} + 4(x-1)^2 y = x+1$

(c) $(x-y)\dfrac{dy}{dx} + y = e^x$

(d) $\dfrac{dy}{dx} = \dfrac{y^2+1}{2xy+1}$

2. Solve

(a) $\dfrac{dy}{dx} - y\tan x = x$

(b) $\dfrac{dx}{dt} + 2tx = 2t^3$

(c) $dx - (\sec y + 2x\tan y)\,dy = 0$

(d) $\dfrac{dy}{dx} = \dfrac{2y}{y-x-y^3}$

(e) $(1+x)\dfrac{dy}{dx} + y = 3x^2$, given $y(0) = 2$.

(f) $2\,dx + (2x+3y)\,dy = 0$, given $y(2) = 0$.

3. A tank initially contains 700 litres of fresh water. A pipe is opened which admits salty water at 10 litres/min. At the same time, a drain is opened to allow 8 litres/min of the mixture to leave the tank. If the inflowing salty water contains 0.01 kg of salt per litre, what is the mass of salt in the tank after 60 minutes? What is the concentration of the salt?

4. Some rocks contain a radioactive isotope of radium, Ra^{226}, which has a half-life of 1590 years and decays into an isotope of lead, Pb^{210}. This lead isotope is itself radioactive, and decays with a half-life of 22 years. Let $R(t)$ be the amount of radium in the rock and $L(t)$ be the amount of lead. Then the rate of change of L is the rate at which lead is produced by the decay of radium, minus the rate at which the lead decays; so $dL/dt = \lambda R - \mu L$ where λ and μ are the decay constants of radium and lead respectively. Given that $R = R_0 e^{-\lambda t}$ and that $L = 0$ at $t = 0$, solve this equation to show that

$$L(t) = \frac{\lambda R_0}{\mu - \lambda}(e^{-\lambda t} - e^{-\mu t}).$$

What are the values of λ and μ?

5. (a) Obtain first-order differential equations that govern the following one-parameter families of curves:

(*i*) $y = Cx^4$;

(*ii*) $\dfrac{x^2}{C} + \dfrac{y^2}{C-1} = 1.$

(b) Find the families of curves that are orthogonal to the families in part (a). Describe the family of curves obtained in the first case. In the second case, the given curves are ellipses when $C > 1$ and hyperbolas when $0 < C < 1$. Show that the ellipses are orthogonal to the hyperbolas. If (x_0, y_0) is any point in the interior of the first quadrant, find the two values of C giving the ellipse and hyperbola through this point.

THE UNIVERSITY OF SYDNEY

MATH1903 INTEGRAL CALCULUS AND MODELLING (ADVANCED)

Semester 2 **Further Exercises Week 10** 2009

1. Find the general solution of

(a) $\dfrac{dx}{dt} - tx = t$ (b) $\dfrac{dy}{dx} = \dfrac{4x^3 - y}{x}$

(c) $\dfrac{dy}{dx} + 2y = e^{-x}$ (d) $x^2\dfrac{dy}{dx} + (1-2x)y = x^2$

2. Find the particular solution of

(a) $\dfrac{dy}{dx} + y\tan x = \sec x, \quad y = 2$ when $x = 0$ (b) $\dfrac{dy}{dx} = \dfrac{2y}{x} + x^4, \quad y = 1$ when $x = 1$

(c) $\dfrac{dx}{dt} + 4x = e^{-4t}\sin 2t, \quad x(0) = 1/2$ (d) $(1+x^2)\dfrac{dy}{dx} + 2xy = 4 + 2x, \quad y(0) = 4$

3. The Howard family borrows \$176,000 to buy a house, and plans to make frequent regular repayments of increasing amounts so that the rate of repayment t years after the start of the loan will be $\$R(t)$/year, where $R(t) = R_0\big(1 + (1/80)t^2\big)$ and R_0 is the initial repayment rate. The interest rate is fixed at 5% per annum, and interest charges are added to the loan amount at frequent regular intervals.

(a) Assuming repayments and interest charges are so frequent that they are effectively continuous, show that the loan amount L varies with time according to the differential equation,

$$\frac{dL}{dt} = \frac{L}{20} - R_0\left(1 + \frac{t^2}{80}\right).$$

(b) Solve this equation, and hence obtain an expression for the amount still owed after t years.

(c) Show that the initial repayment rate R_0 must exceed \$800/year or else the debt will eventually grow out of control.

(d) If $R_0 = \$1000$/year, what is the remaining debt after 20 years?

4. Use the method of partial fractions to evaluate the following integrals of rational functions:

(a) $\displaystyle\int \frac{x}{(x-1)(x-3)}\,dx, \quad 1 < x < 3,$ (b) $\displaystyle\int \frac{x^5}{(x-1)(x-3)}\,dx, \quad 1 < x < 3,$

(c) $\displaystyle\int \frac{6x^4}{x^6 - 1}\,dx, \quad x > 1,$ (d) $\displaystyle\int \frac{2x^2}{x^4+1}\,dx.$

Evaluate the following integrals with the help of suitable substitutions:

(e) $\displaystyle\int \sqrt{\tan\theta}\,d\theta, \quad 0 < \theta < \pi/2,$ (f) $\displaystyle\int \frac{\sqrt{1+y^2}}{y^2}\,dy, \quad y > 0,$

(g) $\displaystyle\int \frac{d\theta}{(a + b\cos\theta)^2}, \quad 0 < \theta < \pi, \ |b| < a,$ (h) $\displaystyle\int \frac{1}{x}\left(\frac{x+1}{x-1}\right)^{2/3} dx, \quad x > 1.$