THE UNIVERSITY OF SYDNEY

MATH1903 INTEGRAL CALCULUS AND MODELLING (ADVANCED)

Semester 2 | **Tutorial Week 11** | 2009

1. (a) Find the general solution for each of the following 2nd-order linear homogeneous DEs:

$$(i)\quad \frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = 0 \quad (ii)\quad \frac{d^2y}{dt^2} + 16y = 0 \qquad (iii)\quad \frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 13y = 0.$$

(b) For each of these DEs, find the particular solution satisfying $y(0) = \dot{y}(0) = -1$.

2. Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0,$$

expressing your answer in terms of sine and cosine functions. Find the particular solution satisfying the boundary values $y(0) = 1$ and $y(\pi/4) = 2$. Is there a solution that satisfies the boundary values $y(0) = 1$ and $y(\pi) = 1$?

3. (From 1995 exam)

(a) Classify each of the following DEs as separable or linear, and find its general solution:

$$(i)\quad \frac{dy}{dx} = \frac{2xy^2}{1+x^2} \qquad (ii)\quad \frac{dy}{dx} = x + \frac{y}{x} \qquad (iii)\quad (x^2+1)\frac{dy}{dx} + x = xy^2$$

(b) Find the particular solution of (iii) for which $y = 0$ when $x = 1$.

(c) Use the substitution $v = xw$ to find the general solution of the differential equation

$$2xv\frac{dv}{dx} = 3v^2 - 4x^2.$$

4. Solve the following differential equations by making suitable substitutions:

(a) $\dfrac{dy}{dx} = \dfrac{y^3}{x(x^2+y^2)}$ (b) $\dfrac{dy}{dx} = \dfrac{1}{x+y}$

(c) $\dfrac{dy}{dx} = \dfrac{x+y}{x+y+2}$ (d) $\dfrac{dy}{dx} = \dfrac{1}{(x+2y)^2+1}$

5. Solve the following differential equations by the methods indicated:

(a) $3x\dfrac{dy}{dx} + y + x^2y^4 = 0$ (a Bernoulli equation, let $w = 1/y^3$).

(b) $\dfrac{dy}{dx} + xy^2 + 3/(4x^3) = 0$ (a Riccati equation, let $y = 1/(2x^2) + 1/w$).

(c) $x\dfrac{dy}{dx} - y = \dfrac{1}{4}\left(\dfrac{dy}{dx}\right)^4$ (a Clairaut equation, differentiate both sides).

6. A tank A contains 100 litres of water in which 5 kilograms of salt has been dissolved. Fresh water enters it at a rate of 2 litres/minute and the resulting mixture, assumed uniform, flows at the same rate into a second tank B which initially contains 50 litres of fresh water. This mixture, also kept uniform, leaves B at the rate of 2 litres/minute. Find the amount of salt in tank B after 50 minutes.

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MATH1903 INTEGRAL CALCULUS AND MODELLING (ADVANCED)

Semester 2 | **Further Exercises Week 11** | 2009

1. Solve the following equations, giving the general solution and then the particular solution $y(x)$ satisfying the given boundary or initial conditions.

(a) $y'' + 4y' + 5y = 0, \quad y(0) = 2,\ y'(0) = 4$

(b) $y'' - 2y' + y = 0, \quad y(2) = 0,\ y'(2) = 1$

(c) $2y'' - 7y' + 5y = 0, \quad y(0) = 1,\ y'(0) = 1$

(d) $y'' + 4y' + 3y = 0, \quad y(-2) = 1,\ y(2) = 1$

(e) $2y'' - 2y' + 5y = 0, \quad y(0) = 0,\ y(2) = 2$

(f) $y'' - 4y' + 4y = 0, \quad y(0) = -2,\ y(1) = 0$

2. Classify the following DEs as separable or linear, and in each case find the general solution for y as an explicit function of x (from 1991/2/3/4 exams):

(a) $\dfrac{dy}{dx} = y^2 \sin x$

(b) $x\dfrac{dy}{dx} + (1-x)y = e^x$

(c) $(1+x^2)\dfrac{dy}{dx} + 4xy = 2x + 2xy^2$

(d) $y\dfrac{dy}{dx} = e^{-y^2} x \sin x$

(e) $x^3\dfrac{dy}{dx} = -3x^2y + \cot x$

(f) $\dfrac{dy}{dx} + 2xy - e^{-x^2}\cot x = 0.$

(g) $e^{-x}\dfrac{dy}{dx} - xy^2 + 3xy = 2x$

(h) $\dfrac{dy}{dx} = \dfrac{5-2y}{1+x^2}$

3. An archaeologist discovers a clay pot full of ashes in an ancient tomb. The ashes contain a radioactive isotope of radium, Ra^{226}, which decays into an isotope of lead, Pb^{210}. This lead isotope is itself radioactive, and also decays.

(*i*) Let $R(t)$ and $L(t)$ respectively denote the amounts of radium Ra^{226} and lead Pb^{210} present in the ashes at a time t years after the fire in which they were formed. Explain briefly why $R(t)$ and $L(t)$ should satisfy the differential equations

(a) $\dfrac{dR}{dt} = -\lambda R$

(b) $\dfrac{dL}{dt} = \lambda R - \mu L$

where λ and μ are the decay constants of Ra^{226} and Pb^{210} respectively.

(*ii*) Let R_0 denote the amount of radium Ra^{226} present in the ashes initially (i.e., immediately after the fire in which they were formed) and suppose it is believed that the ashes initially contained no lead. Solve (*i*)(a) to obtain an expression for $R(t)$. Then solve (*i*)(b) to obtain an expression for $L(t)$.

(*iii*) The half-life of Ra^{226} is 1590 years, while that of Pb^{210} is 22 years. Use this information to determine the values of the decay constants λ and μ.

(*iv*) The ashes are found to contain 90 atoms of Ra^{226} for every atom of Pb^{210}. Deduce the age of the ashes (still assuming that they initially contained no Pb^{210}).