THE UNIVERSITY OF SYDNEY

MATH1903 INTEGRAL CALCULUS AND MODELLING (ADVANCED)

Semester 2 **Tutorial Week 12** 2009

1. Find the general solution of each of the following DEs:

(a) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 5y = 0.$

(b) $\dfrac{d^2x}{dt^2} - 6\dfrac{dx}{dt} + 9x = 0.$

(c) $\dfrac{d^2y}{dt^2} + 9y = 0.$

(d) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 25y = 0.$

2. (a) Find the general solution of each of the following non-homogeneous differential equations:

(i) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x^2$ (ii) $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = 6e^t$ (iii) $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = 6e^{-t}$

(b) For the solutions of (i), (ii), (iii) find the particular solution satisfying the initial conditions

(i) $y(0) = y'(0) = 4.$ (ii) $y(0) = 1$ and $\dot{y}(0) = 0.$ (iii) $y(0) = 2$ and $\dot{y}(0) = 1.$

3. Show that the general solution of the 2nd order linear homogeneous differential equation with constant coefficients such that the auxiliary equation has complex roots $\lambda_1 = \bar{\lambda}_2 = \alpha + i\beta$ can be written in the from $y(x) = re^{\alpha x}\cos(\beta x + \phi)$ with arbitrary real constants r and ϕ.

4. (a) Find the general solution of the non-homogeneous differential equation,

$$\frac{d^2y}{dt^2} + 25y = 100\sin\omega t,$$

and the particular solution subject to the initial conditions $y(0) = 0$ and $\dot{y}(0) = 0$.

(b) For what positive value of ω does your solution break down? What physical phenomenon does this correspond to?

(c) Find the corresponding particular solution of the DE for this special value of ω by two methods:

(i) by the method of undetermined coefficients appropriate to this case (look for short cuts);

(ii) by fixing t in the result of part (a) and taking the limit as ω approaches its special value. You should meet a 0/0-type limit, which can be handled with l'Hôpital's rule.

5. (a) Find the general solution for each of the following higher-order linear equation (the prime denotes d/dx):

(i) $y''' - 3y' + 2y = 0$

(ii) $y''' = 8y$

(iii) $y'''' + 8y'' + 16y = 0$

(iv) $y''' - 3y' + 2y = 12\cosh x$

(b) Find a particular solution of the following differential equations:

(i) $y'' + y' + 2y = 8x^3$

(ii) $y'''' + 3y''' + 10y = 10e^{2x}$

1. (a) Find the general solutions of the following second-order equations:

(*i*) $y'' + y' - 2y = x + 1$ (*ii*) $y'' + 16y = e^x$ (*iii*) $y'' + y' - 2y = e^x$

(*iv*) $y'' - 6y' + 9y = \sin x$ (*v*) $y'' - 6y' + 9y = e^{3x}$.

(b) Find the particular solution of the differential equation $y'' - 6y' + 9y = e^{3x}$ which satisfies the initial conditions $y(0) = 1$ and $y'(0) = 0$.

2. An electrical circuit comprises a capacitor, a resistor, and an inductor connected in series to a voltage generator with a prescribed voltage $V(t)$. In terms of the electric charge $Q(t)$ and the current $I(t) = dQ/dt$, the voltage drops across the components are Q/C for a capacitance C, IR for a resistance R, and $L\,dI/dt$ for an inductance L. The total voltage drop is equal to that supplied by the generator,

$$\frac{Q}{C} + IR + L\frac{dI}{dt} = V(t).$$

(a) Using $I = dQ/dt$, derive the equation,

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \frac{dV}{dt}.$$

(b) Find the general solution of the complementary equation, distinguishing three possible types of behaviour, depending on the values of $R^2 - 4L/C$. Show that, provided $L, R > 0$, all solutions die out as $t \to \infty$.

(c) Find a particular solution for an AC voltage source $V(t) = V_0 \cos(\omega_0 t)$.

3. (a) Show that both $I_1(x) = e^x$ and $I_2(x) = e^{3x}$ are integrating factors for the differential equation,

$$y'' + 4y' + 3y = 4\,\frac{1 - x + x^2}{(1 + x^2)^2}\,e^{-x}.$$

(Note that the method of undetermined coefficients is not appropriate for this equation, so don't attempt to use it. This type of DE will not appear on the exam paper.)

(b) Use the integrating factor $I_1(x)$ to obtain a first integral of the equation in part (a). This first integral will take the form of a linear differential equation of the first order for y in which a constant of integration appears in the coefficients. Then solve the first-order DE to get the general solution of the DE in part (a). (You may get an integral that does not seem elementary—arrange the integrand as the derivative of a product.)

(c) Do the same with the integrating factor $I_2(x)$. Give different names to the constants of integration, e.g., C_3 and C_4.

(d) Obtain y by eliminating y' from the first integrals in parts (b) and (c). This is another way to get the general solution.

(e) Here is a fourth method, known as variation of parameters. Observe that the complementary equation is solved by $y_c = Ae^{-x} + Be^{-3x}$. Attempt a trial solution of the full equation in part (a) of the form $y = A(x)e^{-x} + B(x)e^{-3x}$ and let $A'(x)e^{-x} + B'(x)e^{-3x} = 0$. Substitute into the DE in part (a) and cancel terms to get another constraint involving $A'(x)$ and $B'(x)$. Solve for $A'(x)$ and $B'(x)$ and integrate to get $A(x)$ and $B(x)$, picking up two constants of integration along the way. Deduce the general solution for y. Make sure all versions of the general solution agree (except possibly for the names of the integration constants).