THE UNIVERSITY OF SYDNEY

MATH1903 INTEGRAL CALCULUS AND MODELLING (ADVANCED)

Semester 2	**Tutorial Week 13**	2009

1. Find the general solution of the pair of differential equations,

$$\frac{dx}{dt} = 5x - 3y, \qquad \frac{dy}{dt} = 2x,$$

by differentiating the first equation and then using the second to obtain a linear second-order DE for $x(t)$. (Your solution should have *two* arbitrary constants of integration.) Find the particular solution satisfying the initial conditions $x = 2$, $y = 1$ when $t = 0$.

2. The DE of the previous problem can be written in matrix-vector form as

$$\frac{d}{dt}z = Mz, \quad z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad M = \begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix}.$$

The general solution is given by $z(t) = e^{Mt}z_0$. Compute e^{Mt} explicitly in this example following these steps:

(a) Find the eigenvalues of M, i.e. the roots of the characteristic polynomial $\det(M - \lambda I)$.

(b) Find the eigenvectors of M, i.e. the vectors v_i that satisfy $Mv_i = \lambda_i v_i$.

(c) Form the matrix T whose columns are v_i and verify that $T^{-1}MT = D$ where D is diagonal with eigenvalues λ_1, λ_2 as diagonal entries.

(d) Finally compute $e^{Mt} = Te^{Dt}T^{-1}$ where $e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$.

(e) Verify that for the initial condition $z_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ the solution $e^{Mt}z_0$ is the same as found in the previous problem.

3. The following is a model for an arms race between two superpowers X and Y. Denote the level of preparation of X for war by $x(t)$ and that of Y by $y(t)$, where t represents time. The model consists of the equations,

$$\frac{dx}{dt} = -ax + by, \qquad \frac{dy}{dt} = cx - dy,$$

where a, b, c and d are positive constants.

(a) Eliminate y to obtain a second-order homogeneous linear DE for $x(t)$.

(b) Write down the auxiliary equation of this DE, and its general solution in terms of its roots m_1 and m_2. Show that if $ad > bc$ then both roots of the auxiliary equation must be negative. What does this suggest about the likelihood of war?

(c) Suppose $a = d = 1$, $b = c = 3$, and that $x = 5$, $y = 1$ at the initial time $t = 0$. Find the particular solution for x and y. What can you conclude about the likelihood of war in this case?

(d) Briefly discuss the assumptions underlying the model. Do you think these assumptions are realistic?

1. Consider the system of linear first order equations

$$\dot{z} = Mz$$

where z is a vector and M a square matrix of the same dimension. The general solution is given by $z(t) = e^{Mt}z_0$. In order to compute e^{Mt} we need to compute powers of M. When M can be diagonalised by a matrix T then we have $M^i = TD^iT^{-1}$ where $D = T^{-1}MT$ is diagonal. This problem is about the case when M cannot be diagonalised. For a 2×2 non-diagonalisable matrix M there exist an invertible matrix T such that

$$T^{-1}MT = J \quad \text{where} \quad J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

(a) Derive the formula $M^i = TJ^iT^{-1}$ for powers of M.

(b) Proof by induction that $J^i = \begin{pmatrix} \lambda^i & i\lambda^{i-1} \\ 0 & \lambda^i \end{pmatrix}$.

(c) Use these two results to sum the series of e^{Mt} in this case. Thus show that

$$e^{Mt} = T \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} T^{-1}.$$

(d) Re-derive the result using the following observation: $M = \lambda I + N$ where $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Because I and N commute, $IN = NI$, we can write $e^{(\lambda I + N)t} = e^{\lambda It}e^{Nt}$. Now observe that N is nilpotent, i.e. N^2 is the zero-matrix, and hence e^{Nt} is very simple.

(e) When solved via a 2nd order equation, which case does this correspond to?

2. Consider the pair of differential equations

$$\frac{dx}{dt} = 7x - 2y, \qquad \frac{dy}{dt} = 2x + 3y.$$

(a) Obtain a second-order differential equation for $x(t)$ and find its general solution.

(b) Find the associated general solution for $y(t)$.

(c) Show that, if $x(0) > y(0)$, then $x(t)$ and $y(t)$ increase without limit as $t \to \infty$. Conversely, if $x(0) < y(0)$, show that $x(t)$ and $y(t)$ decrease without limit as $t \to \infty$.

(d) Find the particular solution for $x(t)$ and $y(t)$ satisfying the initial conditions $x(0) = 2$, $\dot{y}(0) = 1$.

(e) Confirm explicitly that the expressions you have obtained for $x(t)$ and $y(t)$ obey the first order equations given in part (a).

3. For the system of two first order equation $\dot{z} = Mz$ with general matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, find a criterion on the matrix M such that the origin is an asymptotically stable equilibrium.