

1. Find the general solution for each of the following differential equations:

$$(i) \quad \frac{dy}{dx} = 1 + \sin x + \sin^2 x, \quad (ii) \quad x^3 \frac{dy}{dx} = 2x^2 + 5, \quad x > 0,$$
$$(iii) \quad \frac{dy}{dx} = \frac{1}{\cosh y}, \quad (iv) \quad \frac{dy}{dx} = \cos y, \quad |y| < \frac{\pi}{2}.$$

In the last case, obtain y explicitly (several equivalent expressions).

2. Find the particular solutions of the following differential equations satisfying the given conditions:

$$(i) \quad \frac{dy}{dx} = 1 - 2x - 3x^2, \quad y = -1 \text{ when } x = 1.$$
$$(ii) \quad e^{2x} \frac{dy}{dx} + 1 = 0, \quad y \rightarrow 2 \text{ as } x \rightarrow \infty.$$
$$(iii) \quad \frac{dy}{dx} = \frac{y}{2} + \frac{1}{2y}, \quad y = 2 \text{ when } x = 0.$$

3. (i) According to Newton's law mass times acceleration equals force. Consider a particle with mass m and velocity v in the field of gravity which exerts a force $-mg$. Using that acceleration is dv/dt write down the differential equation for v and solve it with initial condition $v(0) = 0$.
- (ii) Use the solution of the first part and the fact that $dx/dt = v$ to find the position $x(t)$ when initially $x(0) = h$.
- (iii) Repeat the two previous parts assuming that in addition to the force of gravity there is also a friction force that is proportional to the velocity and opposing the velocity. This is a good model for small velocities. Find the terminal speed $v_\infty = |\lim_{t \rightarrow \infty} v(t)|$ and use v_∞ to eliminate the proportionality constant of the friction term from the solution.
- (iv) Compute the Taylor polynomials $T_3(t)$ of $x(t)$ for the two cases. Hence verify that for small times the solutions with and without friction are close to each other.

1. An upright cylindrical tank of radius R has a small hole of area a at a height x above the base. The speed at which water flows out is given by Torricelli's law as $v = \sqrt{2g(h-x)}$, where h is the height of the water surface above the base. The rate at which the volume of the water in the tank decreases is therefore $va = a\sqrt{2g(h-x)}$.

(a) Use this to show that h satisfies the differential equation,

$$\frac{dh}{dt} = -k\sqrt{h-x},$$

where $k = a\sqrt{2g}/(\pi R^2)$.

(b) Solve to find $h(t)$, given that the water level is initially at $h = H$. How long does it take for the water surface to fall to the level of the hole? How long does it take when the hole is at the base of the tank?

2. According to the Gompertz model, the population N of a colony of animals grows according to the differential equation,

$$\frac{dN}{dt} = \beta N \ln\left(\frac{M}{N}\right),$$

where M is the maximum sustainable population size and β is a positive constant.

(i) Solve this differential equation. (ii) Find $\lim_{t \rightarrow \infty} N(t)$.

(iii) Find the particular solution for which $N = M/2$ when $t = 0$.

3. Denote the solution without friction in Q3(i) by $v_n(t)$ and the solution with friction in Q3(iii) by $v_f(t)$. Show that $|v_n(t)| > |v_f(t)|$ for $t > 0$.

4. This exercise provides more practice in evaluating l'Hôpital-type limits using power series methods. Evaluate

(i) $\lim_{x \rightarrow 0} \frac{e^{-2x^2} - \cos 2x}{x^4},$ (ii) $\lim_{x \rightarrow 0} \{\coth^2 x - \cot^2 x\},$

(iii) $\lim_{x \rightarrow 0} \frac{\ln(1+2x) - 2x\sqrt{1-2x}}{x - \sin x}.$