THE UNIVERSITY OF SYDNEY

Math1903 Integral Calculus and Modelling (Advanced)

Semester 2	**Tutorial Week 9**	2009

1. Find the general solutions of

(a) $(1 + x^2)\dfrac{dy}{dx} + xy = 0$, (b) $x\dfrac{dy}{dx} = y^2 - 1$,

(c) $(x^2y^2 + x^2 + y^2 + 1)\dfrac{dy}{dx} = xy + x$, (d) $ye^x\dfrac{dy}{dx} = y^2 + y - 2$.

2. Find the particular solutions of

(a) $\dfrac{dy}{dx} = xe^{y-x^2}, \quad y(0) = 0$, (b) $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}, \quad y(0) = C$.

[The expression for the solution of (b) can be simplified using the formula $\tan(A + B) = (\tan A + \tan B)/(1 - \tan A \tan B)$, or, equivalently, the identity for inverse tangents from the assignment.]

3. Einstein's Theory of Relativity predicts the existence of black holes: regions in space from which nothing can escape, due to strong gravitational forces. The theory predicts that black holes will be formed when large stars collapse.

However, Einstein's theory did not take into account quantum mechanical effects. In 1975, Stephen Hawking used quantum theory to show that a black hole should glow slightly; that is, it should radiate energy and particles in the same way that a heated object does. Assuming that nothing else falls into the black hole, this causes its mass M to decrease at the rate governed by the differential equation,

$$\frac{dM}{dt} = -\frac{\alpha}{M^2},$$

where t denotes time and α is a constant whose value is not yet known precisely.

(a) Find the general solution $M(t)$ of this differential equation.

(b) Find the particular solution which satisfies the condition that the mass is M_0 when $t = 0$.

(c) How long does it take for a black hole which initially has mass M_0 to lose half its mass? How long does it take for it to evaporate completely?

4. Let y be the number of people in a stable economy who have an income of x or more. The economist Vilfredo Pareto (1848–1923) discovered that the rate at which y decreases with increasing x is directly proportional to the number of people with income x or more and inversely proportional to the income x. Obtain a differential equation for $y(x)$ and solve it to find y in terms of x, given that the minimum income is x_0 and the total population is N.

1. Find the general solutions of

(a) $\dfrac{dy}{dx} = \dfrac{x + \sin x}{3y^2}$, (b) $\dfrac{dx}{dt} = 1 + t - x - tx$, (c) $\dfrac{dy}{dx} = \dfrac{\ln x}{xy + xy^3}$.

2. (a) Find particular solutions satisfying the given conditions for:

(i) $\dfrac{dy}{dx} = \dfrac{1+x}{xy}$ $(x > 0)$, $y(1) = -4$; (ii) $\dfrac{dy}{dt} = \dfrac{ty + 3t}{t^2 + 1}$, $y(2) = 2$.

(b) Find a function $g(x)$ such that $g'(x) = g(x)\big(1 + g(x)\big)$ and $g(0) = 1$.

(c) Find an equation of the curve that passes through the point $(1, 1)$ and whose slope at (x, y) is y^2/x^3.

3. A molecule of substance A can combine with a molecule of substance B to form a molecule of substance X, in a reaction which is denoted $A + B \to X$. According to the Law of Mass Action, the rate of formation of X is proportional to the product of the amounts of A and B present. A test-tube initially contains amounts a and b of substances A and B, respectively, (measured in billions of molecules), but none of substance X.

(a) Let $x(t)$ denote the amount of substance X (i.e., the number of billions of X molecules) produced within the first t seconds. Write down a differential equation for $x(t)$.

(b) Assuming that $a \neq b$, solve this equation to obtain an expression for $x(t)$.

(c) Suppose that initially there are two molecules of B for every molecule of A, and that after 10 seconds there are six molecules of B for every molecule of A. What is the ratio after 30 seconds?

(d) The experiment is repeated, but with the initial amount of substance B halved so as to equal the initial amount a of substance A. (As before, substance X is absent initially.) What fraction of A molecules remain after 30 seconds?

4. [From 1998 exam] According to one model, the growth of a rabbit population on an uninhabited island is described by the differential equation,

$$\frac{dN}{dt} = \alpha\big(1 - \beta\cos 2\pi t\big)N(M - N),$$

where $N(t)$ denotes the population of rabbits after t years, and the constants M, α and β depend upon the size and location of the island.

(a) Explain briefly why you think the $\beta\cos 2\pi t$ term is included in this equation.

(b) Find the general solution of the equation given above.

(c) Show that, according to the model, the ratio $N/(M - N)$ should grow by a factor of $e^{\alpha M}$ over any one-year period.

(d) On a certain island the maximum sustainable population of rabbits is estimated to be 2700. The observed population was 450 on 8th March 1997 and reached 900 exactly one year later. According to the model, what will the population be on 8th March 1999?