The University of Sydney
School of Mathematics and Statistics

Solutions to Sample Quiz 1

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2009

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Quiz 1 covers material up to and including lecture 8. The real quiz will have 10 questions, each worth 1 mark each. There will be answer boxes for you to write your answers in, and *only* your final answers will be marked (1 mark if correct and 0 marks otherwise). Calculators are *not* permitted.

1. Find a closed formula for the lower Riemann sum of $f(x) = e^x$ over the interval $[0, 1]$ using the partition P of $[0, 1]$ into n equal parts.

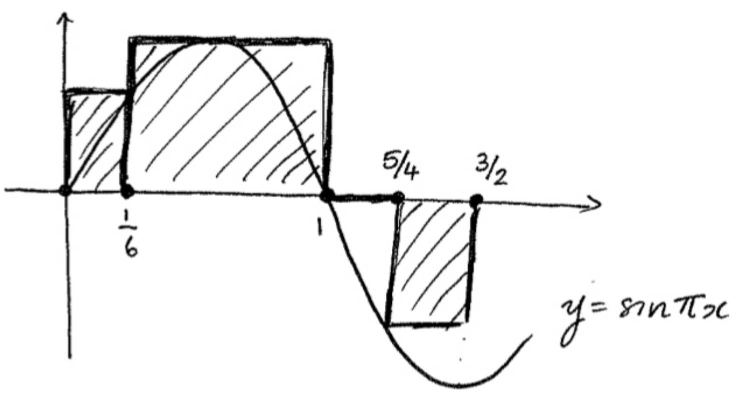
Solution: Since P is the partition of $[0, 1]$ into n equal parts we have $x_j = j/n$ for $j = 0, 1, \ldots, n$. Since $f(x) = e^x$ is monotone increasing, and since we are after the lower Riemann sum, we have $x_j^* = x_{j-1} = (j-1)/n$ for $j = 1, \ldots, n$. Thus the Riemann sum is

$$L_P = \sum_{j=1}^{n} f(x_j^*)\Delta x_j = \frac{1}{n}\sum_{j=1}^{n} e^{(j-1)/n} = \frac{1-e}{n(1-e^{1/n})},$$

where we have used the geometric sum formula.

2. Compute the upper Riemann sum of $f(x) = \sin \pi x$ over the interval $[0, 3/2]$ using the partition $P = \{0, 1/6, 1, 5/4, 3/2\}$.

Solution: The diagram shows the points in the partition P, and the corresponding rectangles that make up the upper Riemann sum.



Thus

$$U_P = \frac{1}{6}\sin\frac{\pi}{6} + \frac{5}{6}\sin\frac{\pi}{2} + \frac{1}{4}\sin\pi + \frac{1}{4}\sin\frac{3\pi}{4} = \frac{11}{12} - \frac{1}{4\sqrt{2}}.$$

3. Given that $f(x) = x \int_0^{2x} te^{-t}\,dt$, find $f''(1)$.

Solution: Using the the product rule, the chain rule, and the Fundamental Theorem of Calculus, we compute

$$\begin{aligned} f'(x) &= \int_0^{2x} te^{-t}\,dt + x\frac{d}{dx}\int_0^{2x} te^{-t}\,dt \\ &= \int_0^{2x} te^{-t}\,dt + x \times 2 \times (2x)e^{-(2x)} = \int_0^{2x} te^{-t}\,dt + 4x^2e^{-2x}. \end{aligned}$$

Then $f''(x) = 2 \times (2x)e^{-(2x)} + 8xe^{-2x} - 8x^2e^{-2x}$, and therefore $f''(1) = 4e^{-2}$.

4. Find the derivative of the function $f(x) = \int_{\sin x}^{3+e^x} \sin t\,dt$.

Solution: Since

$$f(x) = \int_0^{3+e^x} \sin t\,dt - \int_0^{\sin x} \sin t\,dt,$$

the Fundamental Theorem of Calculus and the chain rule imply that

$$f'(x) = e^x \sin(3 + e^x) - \cos x \sin(\sin x).$$

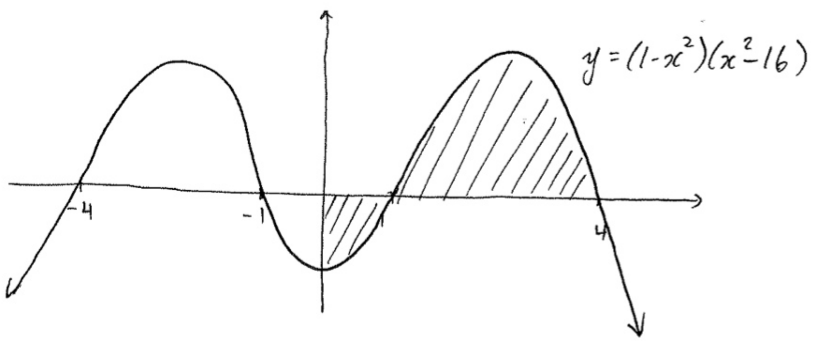
5. Given that $\sin(e^x) = \int_0^x e^t f(t)\,dt$, find $f(x)$.

Solution: By the Fundamental Theorem of Calculus,

$$e^x \cos(e^x) = e^x f(x), \qquad \text{and so} \qquad f(x) = \cos(e^x).$$

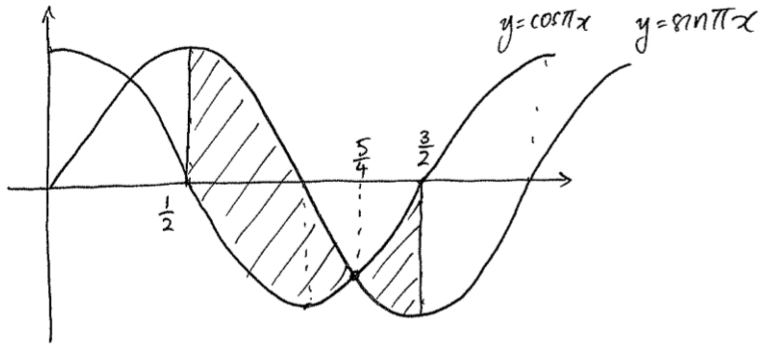
6. Find the value of $x > 0$ which maximises the function $I(x) = \int_0^x (1 - t^2)(t^2 - 16)\,dt$.

Solution: From the picture to see that $x = 4$ gives maximum area.



7. Find the area between the curves $y = \sin \pi x$ and $y = \cos \pi x$ with $1/2 \le x \le 3/2$.

Solution: The sketch is as follows:

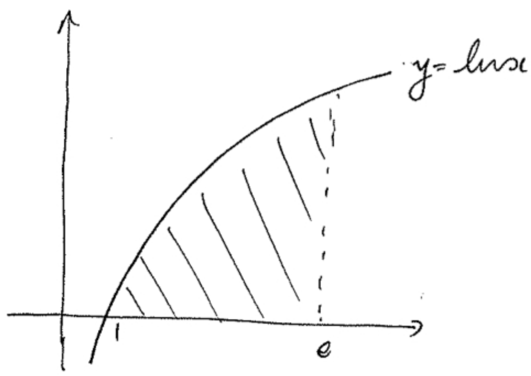


The intersection point in the relevant region is at $x = 5/4$. For $1/2 \le x \le 5/4$ we have $\sin \pi x \ge \cos \pi x$, and for $5/4 \le x \le 3/2$ we have $\cos \pi x \ge \sin \pi x$. Therefore

$$\begin{aligned} A &= \int_{1/2}^{5/4} (\sin \pi x - \cos \pi x)\, dx + \int_{5/4}^{3/2} (\cos \pi x - \sin \pi x)\, dx \\ &= \frac{1}{\pi}(-\cos \pi x - \sin \pi x)\Big|_{1/2}^{5/4} + \frac{1}{\pi}(\sin \pi x + \cos \pi x)\Big|_{5/4}^{3/2} = \frac{2\sqrt{2}}{\pi}. \end{aligned}$$

8. Find the area bounded by the curve $y = \ln x$, the x-axis, and the line $x = e$.

Solution: The picture is:



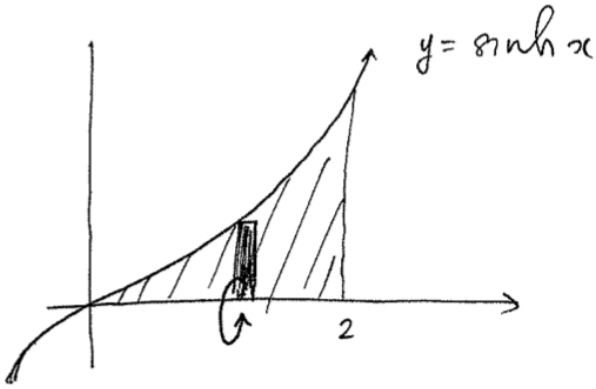
and so the area is

$$A = \int_1^e \ln x\, dx = (x \ln x - x)\Big|_1^e = 1.$$

(Recall: $\int \ln x\, dx$ can be done by integrating by parts, with $u = \ln x$ and $\frac{dv}{dx} = 1$).

9. Compute the volume of the solid obtained by rotating about the x-axis the region bounded by the curve $y = \sinh x$, the x-axis, and the line $x = 2$.

Solution: The picture is:

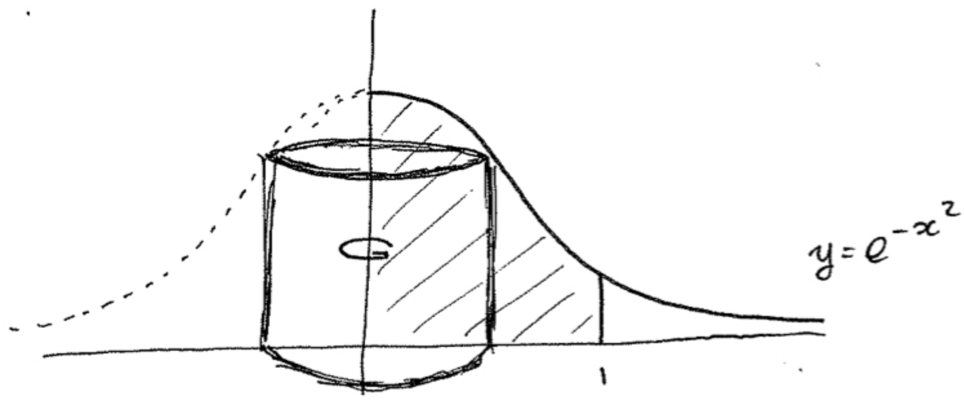


By the disc method,

$$V = \pi \int_0^2 \sinh^2 x \, dx = \frac{\pi}{2} \int_0^2 (\cosh(2x) - 1) \, dx = \frac{\pi}{4} \sinh(4) - \pi.$$

10. Compute the volume of the solid obtained by rotating about the y-axis the region bounded by the curve $y = e^{-x^2}$, the x-axis, the y-axis, and the line $x = 1$.

Solution: The picture is:



By the shell method,

$$V = 2\pi \int_0^1 x e^{-x^2} \, dx = -\pi e^{-x^2} \Big|_0^1 = \pi(1 - e^{-1}).$$

11. Find the length of the curve with parametrisation

$$x(t) = t - \sin t \qquad \text{and} \qquad y(t) = 1 - \cos t \qquad \text{with} \quad t \in [0, 2\pi].$$

Solution: Using the formula for the length of a parametrised curve, the length is

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\,dt \\ &= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t\,dt} \\ &= \sqrt{2}\int_0^{2\pi} \sqrt{1-\cos t}\,dt \\ &= \sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2(t/2)}\,dt \\ &= 2\int_0^{2\pi} \sin(t/2)\,dt \\ &= 8. \end{aligned}$$

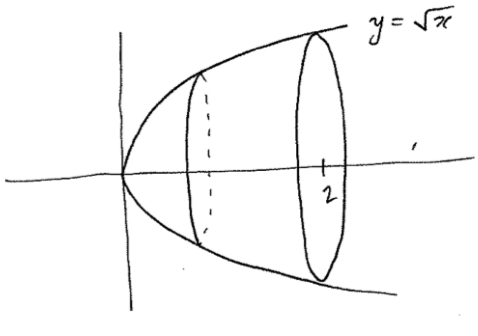
12. Compute the length of the graph $y = \cosh x$ between $x = a$ and $x = b$.

Solution: Using the formula for the length of a graph, the length is

$$L = \int_a^b \sqrt{1 + f'(x)^2}\,dx = \int_a^b \sqrt{1 + \sinh^2 x}\,dx = \int_a^b \cosh x\,dx = \sinh b - \sinh a.$$

13. Compute the surface area of the solid obtained by revolving the part of the graph of $y = \sqrt{x}$ between $x = 0$ and $x = 2$ around the x-axis. Remember to include any end caps.

Solution: The solid looks like



The curved surface area is

$$A_1 = 2\pi \int_0^2 \sqrt{x}\sqrt{1 + \frac{1}{4x}}\,dx = \pi \int_0^2 \sqrt{1+4x}\,dx = \frac{13\pi}{3}.$$

The end cap is a disc of radius $\sqrt{2}$, and therefore has area $A_2 = 2\pi$. Thus

$$A = A_1 + A_2 = \frac{19}{3}\pi.$$

14. Compute the value of the improper integral $\displaystyle\int_0^\infty e^{-x}\cos x\,dx$.

Solution: The improper integral is defined by

$$\int_0^\infty e^{-x}\cos x\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\cos x\,dx.$$

By integrating by parts twice we have

$$\begin{aligned}\int_0^b e^{-x}\cos x\,dx &= e^{-x}\sin x\Big|_0^b + \int_0^b e^{-x}\sin x\,dx\\ &= e^{-b}\sin b + \left(-e^{-x}\cos x\Big|_0^b - \int_0^b e^{-x}\cos x\,dx\right),\end{aligned}$$

and so

$$2\int_0^b e^{-x}\cos x\,dx = 1 + e^{-b}\sin b - e^{-b}\cos b.$$

Hence

$$\int_0^\infty e^{-x}\cos x\,dx = \frac{1}{2}\lim_{b\to\infty}\left(1 + e^{-b}\sin b - e^{-b}\cos b\right) = \frac{1}{2}.$$

15. Decide if the improper integral $\displaystyle\int_0^1 \frac{1}{x^2}\sin\left(\frac{1}{x}\right)\,dx$ exists.

Solution: This improper integral has definition

$$\int_0^1 \frac{1}{x^2}\sin\left(\frac{1}{x}\right)\,dx = \lim_{\epsilon\to 0^+}\int_\epsilon^1 \frac{1}{x^2}\sin\left(\frac{1}{x}\right)\,dx$$

if the limit exists. But

$$\int_\epsilon^1 \frac{1}{x^2}\sin\left(\frac{1}{x}\right)\,dx = \cos\left(\frac{1}{x}\right)\Bigg|_\epsilon^1 = \cos(1) - \cos(1/\epsilon).$$

Since $\displaystyle\lim_{\epsilon\to 0^+}\cos(1/\epsilon)$ does not exist, the integral $\displaystyle\int_0^1 \frac{1}{x^2}\sin\left(\frac{1}{x}\right)\,dx$ does not exist.

16. Decide if the improper integral $\displaystyle\int_0^1 \frac{\cosh x}{\sqrt{x}}\,dx$ exists.

Solution: For $x \in (0,1]$ we have

$$\left|\frac{\cosh x}{\sqrt{x}}\right| \le \frac{\cosh(1)}{\sqrt{x}}.$$

Since $\int_0^1 \frac{\cosh(1)}{\sqrt{x}}\,dx$ exists, so does $\int_0^1 \frac{\cosh x}{\sqrt{x}}\,dx$ (by the Comparison Test).

17. Compute the indefinite integral $\displaystyle\int x^n \ln x\,dx$, where $n \ne -1$.

Solution: Integrating by parts, with $u = \ln x$ and $\frac{dv}{dx} = x^n$ gives

$$\int x^n \ln x\,dx = \frac{x^{n+1}\ln x}{n+1} - \frac{1}{n+1}\int x^n\,dx = \frac{x^{n+1}(\ln x - 1)}{n+1} + C,$$

where C is the constant of integration.

18. Compute the improper integral $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x}}\,dx$.

Solution: The improper integral is computed by taking a limit of proper integrals:

$$\begin{aligned}\int_0^1 \frac{x}{\sqrt{1-x}}\,dx &= \lim_{\epsilon\to 0^+}\int_0^{1-\epsilon}\frac{x}{\sqrt{1-x}}\,dx\\ &= \lim_{\epsilon\to 0^+}\int_1^{\epsilon}\frac{1-u}{\sqrt{u}}(-1)\,du\\ &= \lim_{\epsilon\to 0^+}\int_{\epsilon}^{1}\left(u^{-\frac{1}{2}}-u^{-\frac{1}{2}}\right)du\\ &= \lim_{\epsilon\to 0^+}\left(2-\frac{2}{3}-2\epsilon^{\frac{1}{2}}+\frac{2}{3}\epsilon^{\frac{3}{2}}\right)\\ &= \frac{4}{3}.\end{aligned}$$

19. Change variables to find the indefinite integral $\displaystyle\int \frac{x^2}{\sqrt{1+x^2}}\,dx$, expressing your final answer in terms of x.

Solution: Set $x=\sinh\theta$. Then $dx=\cosh\theta\,d\theta$, and so

$$\int \frac{x^2}{\sqrt{1+x^2}}\,dx = \int \frac{\sinh^2\theta}{\cosh\theta}\cosh\theta\,d\theta = \int \sinh^2\theta\,d\theta.$$

Since $\sinh^2\theta = \frac{1}{2}(\cosh(2\theta)-1)$, the integral is

$$\int \frac{x^2}{\sqrt{1+x^2}}\,dx = \frac{1}{2}\int(\cosh(2\theta)-1)\;d\theta = \frac{1}{4}\sinh(2\theta)-\frac{1}{2}\theta + C.$$

Since $\sinh(2\theta)=2\sinh\theta\cosh\theta = x\sqrt{1+x^2}$ we have

$$\int \frac{x^2}{\sqrt{1+x^2}}\,dx = \frac{1}{2}x\sqrt{1+x^2}-\frac{1}{2}\sinh^{-1}x + C.$$

20. Find a reduction formula for the integral $I_n = \displaystyle\int x^n\cos x\,dx$.

Solution: Integrating by parts twice gives

$$\begin{aligned}\int x^n\cos x\,dx &= x^n\sin x - n\int x^{n-1}\sin x\,dx\\ &= x^n\sin x + nx^{n-1}\cos x - n(n-1)\int x^{n-2}\cos x\,dx.\end{aligned}$$

Therefore

$$I_n = x^n\sin x + nx^{n-1}\cos x - n(n-1)I_{n-2}$$

is the reduction formula.