Sign of a permutation

Given a permutation \( f \) of \( \{1, \ldots, n\} \) its diagram consists of two rows 1, 2, \ldots, \( n \) and each element \( i \) of the first row is connected by a line with \( f(i) \) in the second row. The lines should go from top to bottom and at most two lines should cross at any point. Here is a diagram for the permutation \( f = 41253 \):

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & 2 & 3 & 4 & 5
\end{array}
\]

A permutation is called even or odd respectively if the number of crossings in the diagram is even or odd. The number of crossings depends on how the diagram is drawn but the parity does not. This is explained by the fact that given two diagrams for a permutation, one can be deformed into the other by moving the lines. This deformation can be arranged as a sequence of moves of the following two types:

\[
\begin{array}{c}
\begin{array}{c}
\text{Crosses}
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\text{Simple}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{Simple}
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\text{Crosses}
\end{array}
\end{array}
\]
so that in each case the parity of the number of crossings remains unchanged.

The *sign* $\text{sgn} f$ of a permutation $f$ is defined by

$$
\text{sgn} f = \begin{cases} 
1 & \text{if } f \text{ is even}, \\
-1 & \text{if } f \text{ is odd}.
\end{cases}
$$

This can also be written as

$$
\text{sgn} f = (-1)^{n(f)},
$$

where $n(f)$ is the number of crossings in a diagram for $f$. (This number depends on how the diagram for $f$ is drawn.)

**Theorem.** Let $f$ and $g$ be two permutations of $\{1, \ldots, n\}$. Then for the sign of the permutation $g \circ f$ we have

$$
\text{sgn} (g \circ f) = \text{sgn} g \cdot \text{sgn} f
$$

**Proof.** Let us draw the diagrams for $f$ and $g$ in such a way that the bottom row of the diagram for $f$ coincides with the top row of the diagram for $g$ (here $f = 41253$ and $g = 24135$):
If we ignore the middle row of numbers then this can be regarded as a diagram for the permutation \( g \circ f \). The number of crossings \( n(g \circ f) \) in this diagram is equal to the sum \( n(g) + n(f) \) and so

\[
\sgn (g \circ f) = (-1)^{n(g \circ f)} = (-1)^{n(g)+n(f)} = (-1)^{n(g)} \cdot (-1)^{n(f)} = \sgn g \cdot \sgn f.
\]

The sign of a permutation \( f \) can be equivalently defined by using diagrams of a different kind. Here the top row is the same as before, but the second row is replaced with \( f(1), f(2), \ldots, f(n) \). Each line now connects the same symbol \( i \) in the top and bottom rows. For example, the diagram for \( f = 41253 \) is

A diagram of this type for a permutation \( f \) is in fact a diagram of the first type for the inverse permutation \( f^{-1} \). Indeed, if the \( j \)-th number in the bottom row is \( i \) then \( f(j) = i \). Therefore, \( j = f^{-1}(i) \) and the top number \( i \) is connected with the bottom \( j \)-th number which is \( f^{-1}(i) \).
The theorem implies that
\[ \text{sgn } f \cdot \text{sgn } f^{-1} = \text{sgn } (f \circ f^{-1}) = \text{sgn } \text{Id} = 1. \]

and so, for each permutation \( f \) we have \( \text{sgn } f = \text{sgn } f^{-1} \). This proves that the above two definitions of the sign of a permutation are equivalent.