1. (i) Given languages \( A = \{0, 11\} \) and \( B = \{1, 10, 110\} \) over the alphabet \( \{0, 1\} \), write down \( AB \) and \( BA \).

(ii) Give an example of languages \( A \) and \( B \) over \( \{0, 1\} \) such that \(|AB| \neq |A||B|\).

**Solution**

(i) \( AB = \{01, 010, 0110, 1110, 11110\} \) and \( BA = \{10, 111, 100, 1011, 1100, 11011\} \).

(ii) If \( A = B = \{\varepsilon, 1\} \), then \( AB = \{\varepsilon, 1, 11\} \).

2. Given the alphabet \( \Sigma = \{a, b, c\} \), describe the formal languages designated by the following regular expressions:

(i) \( a + bc \)

(ii) \( a + bc^* \)

(iii) \( a + (bc)^* \)

(iv) \( (ab)^*c^* \)

(v) \( (ab)^*(cb)^* \)

(vi) \( (abca)^* \)

**Solution**

(i) \( \{a, bc\} \).

(ii) \( \{a, b, bc, bcc, bccc, \ldots \} \).

(iii) \( \{\varepsilon, a, bc, bcbc, bcbcbc, \ldots \} \).

(iv) Strings of the form \( abab \ldots abcccc \ldots \).

(v) Strings of the form \( abab \ldots abcabcabcabc \ldots \).

(vi) Strings of the form \( abcaabcaabca \ldots \).

3. Write down a regular expression which designates the following languages

(i) \( \{\varepsilon, b, a, a^2, a^3, a^4, \ldots \} \).

(ii) \( \{\varepsilon, a, b, c, bc, bcbc, bcbcbc, \ldots \} \)

(iii) All words containing \( bca \) over the alphabet \( \{a, b, c\} \).

**Solution**

(i) \( b + a^* \).

(ii) \( a + b + c + (bc)^* \).

(iii) \( (a + b + c)^*bca(a + b + c)^* \).
4. Let $\Sigma = \{a, b, c\}$ and let $A$ be the smallest subset of $\Sigma^*$ such that (a) $\varepsilon \in A$ and (b) if $x \in A$, then $xa \in A$, $xab \in A$ and $xbc \in A$.

(i) Write out the strings of length 2 and 3 in $A$.

(ii) Which of $abaabca$, $bcaababa$, $abbaabca$ is in $A$?

(iii) Let $x_n$ be the number of strings in $A$ of length $n$ and show that $x_n = x_{n-1} + 2x_{n-2}$.

(iv) Find $x_{100}$.

**Solution**

(i) Length 2 strings are $ab$, $aa$, $bc$ and the length 3 strings are $aab$, $abc$, $aba$, $aaa$, $bca$.

(ii) $abaabca \in A$, $bcaababa \in A$, $abbaabca \notin A$.

(iii) Given a string of length $n$ it is either derived from a string of length $n-1$ by adding $a$ or from a string of length $n-2$ by adding $ab$ or $bc$. Thus $x_n = x_{n-1} + 2x_{n-2}$.

(iv) The characteristic equation is $\lambda^2 - \lambda - 2 = 0$ and so $\lambda$ is $-1$ or $2$. Thus $x_n = A(-1)^n + B2^n$ for some $A$ and $B$. From the initial conditions $x_0 = x_1 = 1$ we see that $A = 1/3$ and $B = 2/3$. Thus $x_n = ((-1)^n + 2^{n+1})/3$ and $x_{100} = (1 + 2^{101})/3 = 845100400152152934331135470251$.

5. Prove the following relations in the algebra of regular expressions:

(i) $(r+s)^* = (r^*s)^*r^*$

(ii) $(rs)^* = \varepsilon + r(sr)^*s$

**Solution**

(i) The regular expression $(r+s)^*$ designates all strings formed by $r$ and $s$. On the other hand, every string formed by $r$ and $s$ has the form $r^{k_1}sr^{k_2}sr^{k_3}s \cdots sr^{k_n}$ where $k_1, \ldots, k_n$ are non-negative integers, and so it is contained in the set of strings designated by the regular expression $(r^*s)^*r^*$.

(ii) The regular expression $(rs)^*$ designates the set of strings formed by $\varepsilon$ and $(rs)^k$ with $k \geq 1$. The string $(rs)^k$ can be written as $r(sr)^{k-1}s$ and so, the set $\{(rs)^k \mid k \geq 1\}$ can be designated by the regular expression $r(sr)^*s$. 
1. Let $A = \{00, 11\}$ and $B = \{01, 0\}$. Find the following sets

(i) $AB$  
(ii) $BA$  
(iii) $A^3$  
(iv) $B^2$  
(v) $A^*$  
(vi) $AB^*$

**Solution**

(i) $AB = \{000, 000, 1101, 110\}$.

(ii) $BA = \{0100, 0111, 000, 01\}$.

(iii) $A^3 = \{000000, 000011, 001100, 001111, 110000, 110011, 111100, 111111\}$.

(iv) $B^2 = \{0101, 010, 001, 0\}$.

(v) All strings, including the empty string, built up from 00 and 11 in all possible combinations.

(vi) All strings beginning with 00 or 11 followed by a string made of 0’s and 1’s in which each 1 is preceded by a 0.

2. Determine whether the string 11101 is in the language defined by the following regular expressions.

(i) $(0 + 1)^*$  
(ii) $1^*0^*1^*$  
(iii) $111^*01$  
(iv) $(11)^*(01)^*$

(v) $(111)^*0^*1$  
(vi) $(111 + 000)(00 + 11)$

**Solution**

(i) Yes  
(ii) Yes  
(iii) Yes  
(iv) No

(v) Yes  
(vi) No