THE RING AXIOMS

**Definition.** A ring is a set $R$ with an operation called *addition*: for any $a, b \in R$, there is an element $a + b \in R$, and another operation called *multiplication*: for any $a, b \in R$, there is an element $ab \in R$, satisfying the following axioms:

(i) Addition is associative, i.e.

$$(a + b) + c = a + (b + c)$$

for all $a, b, c \in R$.

(ii) There is an element of $R$, called the *zero element* and written $0$, which has the property that

$$a + 0 = 0 + a = a$$

for all $a \in R$.

(iii) Every element $a \in R$ has a *negative*, an element of $R$ written $-a$, which satisfies

$$a + (-a) = (-a) + a = 0.$$

(iv) Addition is commutative, i.e.

$$a + b = b + a$$

for all $a, b \in R$.

(v) Multiplication is associative, i.e.

$$(ab)c = a(bc)$$

for all $a, b, c \in R$.

(vi) Multiplication is distributive over addition, i.e.

$$a(b + c) = ab + ac$$

and

$$(a + b)c = ac + bc$$

for all $a, b, c \in R$.

**Definition.** A field is a ring $R$ which has the following extra properties:

(vii) $R$ is commutative, i.e. $ab = ba$, $\forall a, b \in R$.

(viii) $R$ has a nonzero identity element $1$.

(ix) Every nonzero element of $R$ is invertible.

**Definition.** An integral domain is a ring $R$ which satisfies the following extra properties:

(vii) $R$ is commutative.

(viii) $R$ has a nonzero identity element $1$.

(ix)’ $R$ has no zero divisors.