Now for something completely different. Euclid’s Algorithm.

We translated

\[
\begin{align*}
911 &= 1.671 + 240 \\
671 &= 2.240 + 191
\end{align*}
\]

to

\[
\begin{align*}
\frac{911}{671} &= 1 + \frac{240}{671} \\
\frac{671}{240} &= 2 + \frac{191}{240}
\end{align*}
\]

and so on.

Now I want to read this again in a different way. We can say that \(\frac{911}{671} = 1 + \frac{240}{671}\) can become

Every positive rational number is an integer \(k\) plus a fraction \(f\) between 0 and 1.

And then:

One over the fraction is an integer plus a fraction.

This is now a restatement of Euclid’s Algorithm. It gives it in a way that can extend to the positive real numbers.

Every positive real number is an integer \(k\) plus a ‘fraction’ \(f\) between 0 and 1.

And then:

One over the ‘fraction’ is an integer plus a ‘fraction’.

Here I am loosely reading the word fraction so that it applies to all positive real numbers. For example \(\pi = 3.1415926535\ldots = 3 + 0.1415926535\ldots\)

Example: Now I want us to use our calculators to apply Euclid’s Algorithm in this new form to \(\sqrt{7}\).

\[
\sqrt{7} = 2 + 0.645751311064591\ldots
\]

\[
\begin{align*}
1 \\
0.645751\ldots
\end{align*}
\]

\[
\begin{align*}
1 \\
0.548583\ldots
\end{align*}
\]

\[
\begin{align*}
1 \\
0.822875\ldots
\end{align*}
\]

\[
\begin{align*}
1 \\
0.25250\ldots
\end{align*}
\]

\[
= 4 + 0.645751311064616\ldots
\]
The calculator is *trying* to tell us that the process has repeated, but of course because calculators round off somewhere all such numbers, at each step they make mistakes, which accumulate slowly. It is clear that

\[ \sqrt{7} = [2, 1, 1, 1, 4]. \]

The Magic Table gives us:

<table>
<thead>
<tr>
<th>2</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>4</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>33</td>
<td>41</td>
<td>115</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>14</td>
<td>17</td>
<td>31</td>
</tr>
</tbody>
</table>

The convergents \(2, \frac{5}{2}, \frac{33}{14}\) and so on, are very good approximations to \(\sqrt{7}\), In fact they differ from \(\sqrt{7}\) by at most \(\frac{1}{q_n}\) for each \(n\).

We need to be able to do this calculation *exactly*, avoiding all problems with round-off errors. We’ll do this with

**Example: \(\sqrt{13}\)**

\[
\sqrt{13} = 3 + \sqrt{13} - 3 \\
\frac{1}{\sqrt{13} - 3} = \frac{\sqrt{13} + 3}{4} = 1 + \frac{\sqrt{13} - 1}{4} \\
\frac{4}{\sqrt{13} - 1} = \frac{4(\sqrt{13} + 1)}{12} = \frac{\sqrt{13} + 1}{3} = 1 + \frac{\sqrt{13} - 2}{3} \\
\frac{3}{\sqrt{13} - 2} = \frac{3(\sqrt{13} + 2)}{9} = \frac{\sqrt{13} + 2}{3} = 1 + \frac{\sqrt{13} - 1}{3} \\
\frac{3}{\sqrt{13} - 1} = \frac{3(\sqrt{13} + 1)}{12} = \frac{\sqrt{13} + 1}{4} = 1 + \frac{\sqrt{13} - 3}{4} \\
\frac{4}{\sqrt{13} - 3} = \frac{4(\sqrt{13} + 3)}{4} = 6 + (\sqrt{13} - 3),
\]

and the process has repeated. Thus

\[ \sqrt{13} = [3, 1, 1, 1, 1, 6]. \]

\[
3 \quad 1 \quad 1 \quad 1 \quad 1 \quad 6 \quad 1 \quad 1 \quad 1 \quad 1 \quad 6 \\
\sqrt{13} : 0 \quad 1 \quad 3 \quad 4 \quad 7 \quad 11 \quad 18 \quad 119 \quad 137 \quad 256 \quad 393 \quad 649 \\
1 \quad 0 \quad 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 33 \quad 38 \quad 71 \quad 109 \quad 180
\]

\[
2 \quad 1 \quad 1 \quad 1 \quad 4 \quad 1 \quad 1 \quad 1 \quad 4 \\
\sqrt{7} : 0 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 33 \quad 41 \quad 74 \quad 115 \\
1 \quad 0 \quad 1 \quad 1 \quad 2 \quad 3 \quad 14 \quad 17 \quad 31 \quad 48
\]

Remark: If we look at the pair of integers in the column just prior to the even integer which indicates the part of the continued fraction which repeats (4 for \(\sqrt{7}\) and 6 for \(\sqrt{13}\), we find
integers which solve the so-called Pell’s Equation, \( x^2 - dy^2 = \pm 1 \). In fact
\[
\begin{align*}
8^2 - 7.3^2 &= 1 \\
127^2 - 7.48^2 &= 1 \\
18^2 - 13.5^2 &= -1 \\
649^2 - 13.180^2 &= 1 \\
842401^2 - 13.23640^2 &= 1 \\
709639444801^2 - 13.393637139280^2 &= 1
\end{align*}
\]
and so on.

**Remark:** It is a fact that \( x^2 - dy^2 = 1 \) always has infinitely many integer solutions for any non-square integer \( d \) - and you find them in the Magic Table, though I won’t prove that here.

It is an interesting fact that Fermat wrote to Mersenne that he could solve the equations \( x^2 - 61y^2 = 1 \) and \( x^2 - 109y^2 = 1 \). He knew that the smallest non-zero integers which solve those equations are:

\[
\begin{align*}
61 \ x &= 1766913509 \ y = 302253980 \\
109 \ x &= 158070671986249 \ y = 17030517780364
\end{align*}
\]

Remark (History of Civilisation)
\[
\pi = [3, 7, 15, 1, 292, \ldots]
\]

\[
\begin{array}{cccccc}
3 & 7 & 15 & 1 & 292 \\
0 & 1 & 3 & 22 & 333 & 355 & 103993 \\
1 & 0 & 1 & 7 & 106 & 113 & 33102
\end{array}
\]

Kings 1, 7, 23 And he made a molten sea of 10 cubits from brim to brim and it was evenly round about and five cubits the height thereof and a line of thirty cubits did compass it about.

The approximation \( \frac{22}{7} \) was used in school until recently when calculators we able to give better (decimal) approximations. Of course, \( \frac{22}{7} \) is a very good approximation to \( \pi \).

The approximation \( \frac{355}{113} \) was known to Chinese mathematicians in the 4th or 5th century AD.

\[
\pi = [3, 7, 15, 1, 292, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1, 4, 2, 6, 6, 1, \ldots]
\]