We need to check that infinite simple continued fractions converge to unique real numbers, otherwise they are meaningless objects.

To show you that this is the case in a different situation, consider the following array:

\[
1 = 3.7 + 10(-2), \text{ and so } \\
\frac{1}{3} = 7 + 10 \cdot -\frac{2}{3}. \text{ Also } \\
-2 = 3.6 + 10(-2) \text{ and so } \\
\frac{-2}{3} = 6 + 10 \cdot -\frac{2}{3}. \text{ Hence } \\
\frac{1}{3} = 7 + 60 + 10^2 \cdot -\frac{2}{3}, \\
= 667 + 10^3 \cdot -\frac{2}{3}, \\
= 6667 + 10^4 \cdot -\frac{2}{3} \\
\vdots
\]

Taking the limit (if that were possible), we would have

\[
\frac{1}{3} = \ldots 6667.
\]

This is not meaningful in ordinary arithmetic, but, we could argue as follows:

Let \( x = \ldots 6667 \). Then \( x - 7 = \ldots 66660 \) and so

\[
\frac{x - 7}{10} = \ldots 6666.
\]

Hence

\[
\frac{x - t}{10} + 1 = \ldots 66667 = x.
\]

We can solve that to get \( x = \frac{1}{3} \).

Now suppose that we have a sequence \( a_1, a_2, a_3, \ldots \) of positive integers and we use this to produce a Magic Table and a sequence of integers \( p_n, q_n \) and rational numbers \( \frac{p_n}{q_n} \). We will show that as \( n \to \infty \), \( \frac{p_n}{q_n} \) approaches a unique real number, and we will define the simple continued fraction

\[
[a_1, a_2, a_3, \ldots] = \lim_{n \to \infty} \frac{p_n}{q_n}.
\]
Theorem 5(iii) shows that
\[ \frac{p_n}{q_n} = \frac{p_1}{q_1} + \frac{1}{q_1q_2} - \frac{1}{q_2q_3} + \ldots + (-1)^n \frac{1}{q_{n-1}q_n}. \]

Theorem 5(ii) shows that
\[ p_{\ell-2}q_{\ell} - q_{\ell-2}p_{\ell} = (-1)^\ell a_\ell \]
and so
\[ \frac{p_{\ell-2}}{q_{\ell-2}} - \frac{p_\ell}{q_\ell} = (-1)^\ell \frac{a_\ell}{q_{\ell-2}q_{\ell}}. \]

Now if we take \( \ell = 3 \) we get
\[ \frac{p_1}{q_1} - \frac{p_3}{q_3} = -\frac{a_3}{q_2q_3} < 0 \]
and so
\[ \frac{p_1}{q_1} < \frac{p_3}{q_3}. \]

But then taking \( \ell = 5 \), we get
\[ \frac{p_3}{q_3} - \frac{p_5}{q_5} = -\frac{a_5}{q_4q_5} < 0 \]
and so
\[ \frac{p_3}{q_3} < \frac{p_5}{q_5}. \]

It follows that
\[ \frac{p_1}{q_1} < \frac{p_3}{q_3} < \frac{p_5}{q_5} < \ldots. \]

Similarly, taking first \( \ell = 4 \) and then \( \ell = 6 \) we get
\[ \ldots < \frac{p_6}{q_6} < \frac{p_4}{q_4} < \frac{p_2}{q_2}. \]

Now if \( \frac{p_{2m}}{q_{2m}} < \frac{p_{2n+1}}{q_{2n+1}} \) then if \( m < n \), we have
\[ \frac{p_{2n}}{q_{2n}} < \frac{p_{2m}}{q_{2m}} < \frac{p_{2n+1}}{q_{2n+1}}. \]

But by Theorem 5(i),
\[ \frac{p_{2n}}{q_{2n}} - \frac{p_{2n+1}}{q_{2n+1}} > 0. \]

This is a contradiction.

Now suppose that \( m \geq n \). Then we have
\[ \frac{p_{2m}}{q_{2m}} < \frac{p_{2n+1}}{q_{2n+1}} < \frac{p_{2m+1}}{q_{2m+1}}. \]

But this is the same problem since
\[ \frac{p_{2m}}{q_{2m}} - \frac{p_{2m+1}}{q_{2m+1}} = (-1)^{2m} > 0. \]

Hence it follows that every \( \frac{p_{even}}{q_{even}} \) is greater than every \( \frac{p_{odd}}{q_{odd}} \).
Hence we have

\[
\frac{p_1}{q_1} < \frac{p_3}{q_3} < \frac{p_5}{q_5} < \ldots < \frac{p_6}{q_6} < \frac{p_4}{q_4} < \frac{p_2}{q_2}.
\]

Since

\[
\left| \frac{p_{2n}}{q_{2n}} - \frac{p_{2n+1}}{q_{2n+1}} \right| = \frac{1}{q_{2n}q_{2n-1}} \leq \frac{1}{2n(2n-1)} \to 0
\]

as \( n \to \infty \), it follows that \( \frac{p_n}{q_n} \) has a unique limit as \( n \to \infty \) and we have defined a unique real number

\[
x = [a_1, a_2, a_3, \ldots].
\]

**Theorem 7.** Every irrational real number \( r > 0 \) can be approximated by infinitely many rational numbers \( \frac{p_n}{q_n} \) such that \( |r - \frac{p_n}{q_n}| < \frac{1}{q_n^2} \).

Proof: Apply Euclid’s Algorithm to \( r \) and obtain infinitely many positive integer partial quotients \( a_1, a_2, \ldots \).

Now draw up the Magic Table from these infinitely many partial quotients and get infinitely many convergents \( \frac{p_n}{q_n} \) which all satisfy \( |r - \frac{p_n}{q_n}| < \frac{1}{q_n^2} \). This completes the proof.

It is an extremely interesting fact that this is not true of rational numbers. One cannot approximate rational numbers in this way, at least not with infinitely many such approximations.

**Remark:** (not quite a proof) A number is rational if and only if it has a finite simple continued fraction.

For a finite simple continued fraction is obviously a positive rational number, and conversely if we have \( \frac{p}{q} \) a rational number, then Euclid’s Algorithm applied to \( (p, q) \) gives a finite continued fraction for \( \frac{p}{q} \).

**Example 1.** \( x = [2, 1, 3, 5] \)

\[
\begin{array}{cccccc}
2 & 1 & 3 & 5 \\
0 & 1 & 2 & 3 & 11 & 58 \\
1 & 0 & 1 & 1 & 4 & 21
\end{array}
\]

and \( x = \frac{58}{21} \).

**Example 2.** \( x = [2, 1, 3] \).
\[
x = 2 + \frac{1}{1 + \frac{1}{3 + \frac{1}{1 + \frac{1}{3 + \ldots}}}}
\]
\[
\frac{1}{x - 2} = 1 + \frac{1}{3 + \frac{1}{1 + \frac{1}{3 + \ldots}}}
\]
\[
\frac{1 - x + 2}{x - 2} = 3 + \frac{1}{1 + \frac{1}{3 + \frac{1}{1 + \ldots}}}
\]
\[
\frac{x - 2}{3 - x} - 3 = \frac{1}{1 + \frac{1}{3 + \ldots}} = x - 2.
\]

Hence \(x - 2 - 9 + 3x = (x - 2)(3 - x) = -6 + 5x - x^2\) and so \(x^2 - x - 5 = 0\).

Hence
\[
x = \frac{1 \pm \sqrt{1 + 20}}{2} = \frac{1 \pm \sqrt{21}}{2}.
\]

Since \(\frac{1 - \sqrt{21}}{2} < 0\) and \(x\) is clearly positive, we must have
\[
[2, 1, 3] = \frac{1 + \sqrt{21}}{2}.
\]

Verify this.