1. Consider the systems of congruences
   \[
   \begin{align*}
   & (i) \quad x \equiv 102 \pmod{245} \\
   & (ii) \quad x \equiv 387 \pmod{875}
   \end{align*}
   \]
   Use CRT to write each system of congruences as an equivalent system modulo prime powers. Hence decide if there is a solution or not.

2. \( (i) \) Show that \( x^2 - 2 \) and \( x^2 + 3x - 1 \) are both prime in \( \mathbb{Z}_{11}[x] \).
   \( (ii) \) Show that \( \mathbb{Z}_{11}[x]_{x^2 - 2} \cong \mathbb{Z}_{11}[y]_{y^2 + 3y - 1} \), by finding \( a, b \in \mathbb{Z}_{11} \) such that \( x \rightarrow a + by \) extends to an isomorphism.

3. Write down the square modulo 23 and use this fact to evaluate the Legendre symbol \( \left( \frac{a}{23} \right) \) for every \( a \in \mathbb{Z}_{23} \). For every square \( a \in \mathbb{Z}_{23} \) write down \( \sqrt{a} \in \mathbb{Z}_{23} \).

4. Show that the sum of two squares modulo a prime \( p \) may or may not be a square. Similarly show that the sum of two non-squares may or may not be a non-square.

5. Apply the method of Gauss’s Lemma to evaluate
   \[
   \left( \frac{-5}{13} \right), \left( \frac{14}{23} \right), \left( \frac{3}{73} \right).
   \]

6. \( (i) \) Given that 5 is a generator for \( \mathbb{Z}_{73}^* \) find an element \( h \) such that
   \[ h^8 = 1. \]
   \( (ii) \) Let \( x = h + h^{-1} \) Prove that \( x^2 = 2 \) without using the fact that you know what \( h \) is. Hence write down \( \sqrt{2} \in \mathbb{Z}_{73} \) in terms of \( h \).
   \( (iii) \) Now calculate \( \sqrt{2} \in \mathbb{Z}_{73} \) and also \( \left( \frac{2}{73} \right) \).

7. (A bit harder) Let \( \omega \neq 1 \) be a complex cube root of 1. Show that \( \omega = \frac{-1 \pm \sqrt{-3}}{2} \) and that \( 1 + \omega + \omega^2 = 0 \).
   You may suppose that \( \omega = \frac{-1 + \sqrt{-3}}{2} \) and assume that the arithmetic in \( \mathbb{Z}[\omega] \) is exactly similar to arithmetic in \( \mathbb{Z}[i] \), \( \mathbb{Z}_p[x] \) and \( \mathbb{Z} \). Define \( N(a + b\omega) = |a + b\omega|^2 = a^2 - ab + b^2 \).
   \( (i) \) Show that \( a + b\omega \) is a unit in \( \mathbb{Z}[\omega] \) if and only if \( N(a + b\omega) = 1 \).
   \( (ii) \) Show that there are exactly 6 units in \( \mathbb{Z}[\omega] \), namely \( \pm 1, \pm \omega, \pm \omega^2 = \pm(-1 - \omega) \).
   \( (iii) \) Prove that 3 is not a prime in \( \mathbb{Z}[\omega] \) by exhibiting a proper factorization of it.
   \( (iv) \) Show that 2 is prime in \( \mathbb{Z}[\omega] \).