THE UNIVERSITY OF SYDNEY

Differential Equations & Biomathematics

| Semester 1 | **Assignment 1** | 2011 |
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This assignment is due at 5pm on Thursday 19th May at room 626.

**1.** In some diseases, immunity is lost over time and so members of the removed class return to the susceptible class. A simple model for this type of disease dynamics is given by

$$\begin{aligned}\frac{dx}{dt} &= -\beta xy + \alpha z\\ \frac{dy}{dt} &= \beta xy - \gamma y\\ \frac{dz}{dt} &= \gamma y - \alpha z.\end{aligned}$$

*(i)* Draw a flow diagram which illustrates the movement of individuals between the susceptible, infectives and removed classes. Label the arrows in your diagram with the appripriate rate constants.

*(ii)* By adding the equations together, show that $N = x + y + z$ is constant. Hence write the system of equations above as a system of two equations of the form

$$\begin{aligned}\frac{dx}{dt} &= f(x, y)\\ \frac{dy}{dt} &= g(x, y).\end{aligned}$$

*(iii)* Show that these equations have two steady states and find their linear stability. For each steady state, note any restrictions on parameters that are needed for the steady state to exist.

*(iv)* Sketch the phase plane for this system when both steady states exist.

*(v)* How does the outcome of the this model compare to the basic SIR model described in lectures? What insight does the model given into the behaviour of diseases which give only temporary immunity? What insight does it give into how to control such diseases and minimise their spread?

**2.** Two species which have population densities $u$ and $v$ respectively, interact according to the model

$$\begin{aligned}\frac{du}{dt} &= ru\left(1 - \frac{u}{K}\right) - \alpha uv\\ \frac{dv}{dt} &= sv - \beta uv.\end{aligned}$$

*(i)* What type of interaction exists between these two species?

*(ii)* Show that if $\beta K > s$, there will be three steady states and that only one of these steady states is linearly stable.

*(iii)* Sketch the phase plane when $\beta K > s$ and when $\beta K < s$.

*(iv)* What is the outcome of this model? Is this a good model? Why or why not?

**3.** (*i*) Sketch the phase plane for the system

$$\frac{dx}{dt} = 3x^2 + 3y^2 - 6x$$
$$\frac{dy}{dt} = 6xy - 6y.$$

Hint: the $\dot{x} = 0$ nullcline can be written in the form $(x+a)^2 + y^2 = b^2$.

(*ii*) Show that this system is a gradient system and find the underlying gradient function $F(x, y)$.

(*iii*) Hence draw the level curves of $z = F(x, y)$.

**4.** Find and use a suitable Liapunov function $V(x, y)$ to show that $(0, 0)$ is an asymptotic steady state of

$$\frac{dx}{dt} = -x^3 + y^4$$
$$\frac{dy}{dt} = -y^3 - 3xy^3.$$