MATH3063 – Assignment 2 – 2011.

1. (i)

$$\begin{tikzcd} \boxed{\begin{matrix} S \\ (x) \end{matrix}} \arrow[r, "\beta"'] & \boxed{\begin{matrix} I \\ (y) \end{matrix}} \arrow[r, "\gamma"'] & \boxed{\begin{matrix} R \\ (z) \end{matrix}} \arrow[ll, bend right, "\alpha"'] \end{tikzcd}$$

(ii)

$$\frac{dx}{dt} = -\beta x y + \alpha z$$

$$\frac{dy}{dt} = \beta x y - \gamma y$$

$$\frac{dz}{dt} = \gamma y - \alpha z$$

Adding these equations we get

$$\frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} = 0.$$

or $$\frac{d(x+y+z)}{dt} = \frac{dN}{dt} = 0$$

Hence $N = x + y + z = \text{constant}$

Substitute $z = N - x - y$ into the $\frac{dx}{dt}$ eqn.

So $$\frac{dx}{dt} = -\beta x y + \alpha(N - x - y) = f(x, y)$$

$$\frac{dy}{dt} = \beta x y - \gamma y = g(x, y).$$

(iii) At steady state $\frac{dx}{dt} = \frac{dy}{dt} = 0$

$$-\beta x y + \alpha N - \alpha x - \alpha y = 0.$$

$$y(\beta x - \gamma) = 0$$

so $y = 0$ or $x = \gamma/\beta$.

When $y = 0$ $x = N$

When $x = \gamma/\beta$ $\quad -\gamma y + \alpha N \quad \frac{\alpha\gamma}{\beta} \quad \alpha y = 0$

$$(\alpha+\gamma) y = \alpha(N - \gamma/\beta).$$

$$y = \frac{\alpha}{\alpha+\gamma}(N - \gamma/\beta).$$

Hence there are two steady states, $(N, 0)$ and $\left(\gamma/\beta, \frac{\alpha}{\alpha+\gamma}(N - \gamma/\beta)\right)$. The second only exists if $N - \gamma/\beta > 0$ or $N > \gamma/\beta$.

Linear Analysis

$$J = \begin{pmatrix} -\beta y - \alpha & -\beta x - \alpha \\ \beta y & \beta x - \gamma \end{pmatrix}.$$

$$J(N, 0) = \begin{pmatrix} -\alpha & -\beta N - \alpha \\ 0 & \beta N - \gamma \end{pmatrix}.$$

$$\text{Det } J = -\alpha\beta(N - \gamma/\beta) \begin{cases} < 0 & \text{if } N > \gamma/\beta \\ > 0 & \text{if } N < \gamma/\beta \end{cases}$$

So $(N,0)$ is a saddle if the nonzero steady state exists (i.e. when $N > \gamma/\beta$).

Further investigation is needed when $N < \gamma/\beta$.

$\mathrm{Tr}\, J = -\alpha + \beta N - \gamma = \frac{1}{\beta}\left(N - \gamma/\beta - \frac{\alpha}{\beta}\right)$.

Now $N - \gamma/\beta < 0$ and $\alpha > 0$ so $\mathrm{Tr}\, J < 0$.

So when $N < \gamma/\beta$, $(N, 0)$ is a stable node or focus.

$$(\mathrm{Tr}\, J)^2 - 4 \,\mathrm{Det}\, J = \beta^2\left(N - \frac{\gamma}{\beta} - \frac{\alpha}{\beta}\right)^2 + 4\alpha\beta\left(N - \gamma/\beta\right)$$

$$= \beta^2\left(\left(N - \frac{\gamma}{\beta}\right)^2 - 2\frac{\alpha}{\beta}\left(N - \frac{\gamma}{\beta}\right) + \frac{\alpha^2}{\beta^2} + 4\frac{\alpha}{\beta}\left(N - \gamma/\beta\right)\right)$$

$$= \beta^2\left(\left(N - \frac{\gamma}{\beta}\right) + \frac{\alpha}{\beta}\right)^2$$

$$> 0$$

So $(N, 0)$ is a stable node when $N < \gamma/\beta$.

(iv) Nullclines: $\dot{x} = 0$ $(\beta x + \alpha) y = \alpha N - \alpha x$

$$y = \frac{\alpha (N - x)}{\alpha + \beta x}.$$

$\dot{y} = 0$ $y = 0$ or $x = \gamma/\beta$

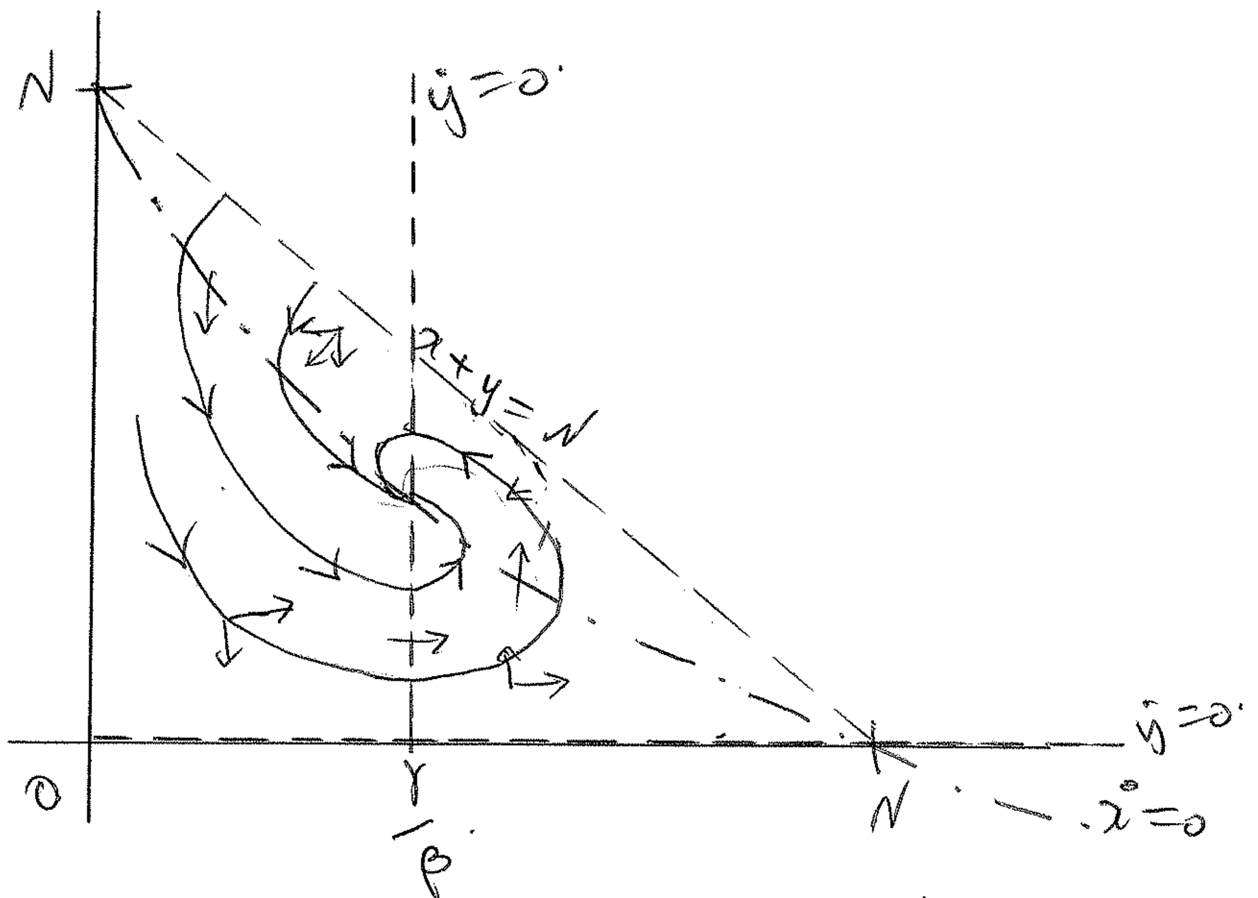
Flows $\dot{x} \gtrless 0$ if $-\beta x y + \alpha N - \alpha x - \alpha y \gtrless 0$ $\rightleftarrows$

or $(\beta x + \alpha) y \lessgtr \alpha N - \alpha x$

$$y \lessgtr \frac{\alpha N - \alpha x}{\beta x + \alpha} \text{ as } \beta x + \alpha > 0$$

$\dot{y} \gtrless 0$ if $x \gtrless \gamma/\beta$ $\uparrow \downarrow$.

If there are two steady states then $N > \gamma/\beta$.
The relevant part of the phase plane is
$x \geq 0$, $y \geq 0$ and $x + y \leq N$.



In this model there are always infective individuals present in the population, unlike standard SIR models where the disease dies out as $t \to \infty$. The disease can be controlled by increasing γ/β by either increasing the removal rate γ or by decreasing the contact rate β.

2. (i) This is a competition interaction, as each species reduces the growth rate of the other.

(ii) At steady state

$$u\left(r\left(1-\frac{u}{k}\right)-\alpha v\right)=0$$

$$v(s-\beta u)=0$$

So steady states are $(0,0)$, $(k,0)$ and $\left(\frac{s}{\beta}, \frac{r}{\alpha}\left(1-\frac{s}{\beta k}\right)\right)$, provided $\beta k > s$.

$$J=\begin{pmatrix} r(1-u/k)-\alpha v-\frac{ru}{k} & -\alpha u \\ -\beta v & s-\beta u \end{pmatrix}.$$

$$J(0,0)=\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$$ Characteristic eqn $(\lambda-r)(\lambda-s)=0$ so $\lambda=r,s>0$ and $(0,0)$ is an unstable node.

$$J(k,0)=\begin{pmatrix} -r & -\alpha k \\ 0 & s-\beta k \end{pmatrix}$$

Characteristic equation is $(-r-\lambda)(s-\beta k-\lambda)=0$ so $\lambda=-r,\ s-\beta k$. If $\beta k>s$, these are both negative so $(k,0)$ is a stable node.

$$J\left(\frac{s}{\beta}, \frac{r}{\alpha}\left(1-s/\beta k\right)\right)=\begin{pmatrix} -\frac{rs}{\beta k} & -\frac{\alpha s}{\beta} \\ -\beta v^* & 0 \end{pmatrix}$$

where $v^*=\frac{r}{\alpha}\left(1-s/\beta k\right)$

$\text{Det } J = -\alpha\beta v^* s/\beta < 0$

So $\left(\frac{s}{\beta}, \frac{r}{a}\left(1-\frac{s}{\beta K}\right)\right)$ is a saddle.

Therefore, of the three steady states, only $(K, 0)$ is linearly stable.

(iii) Nullclines:

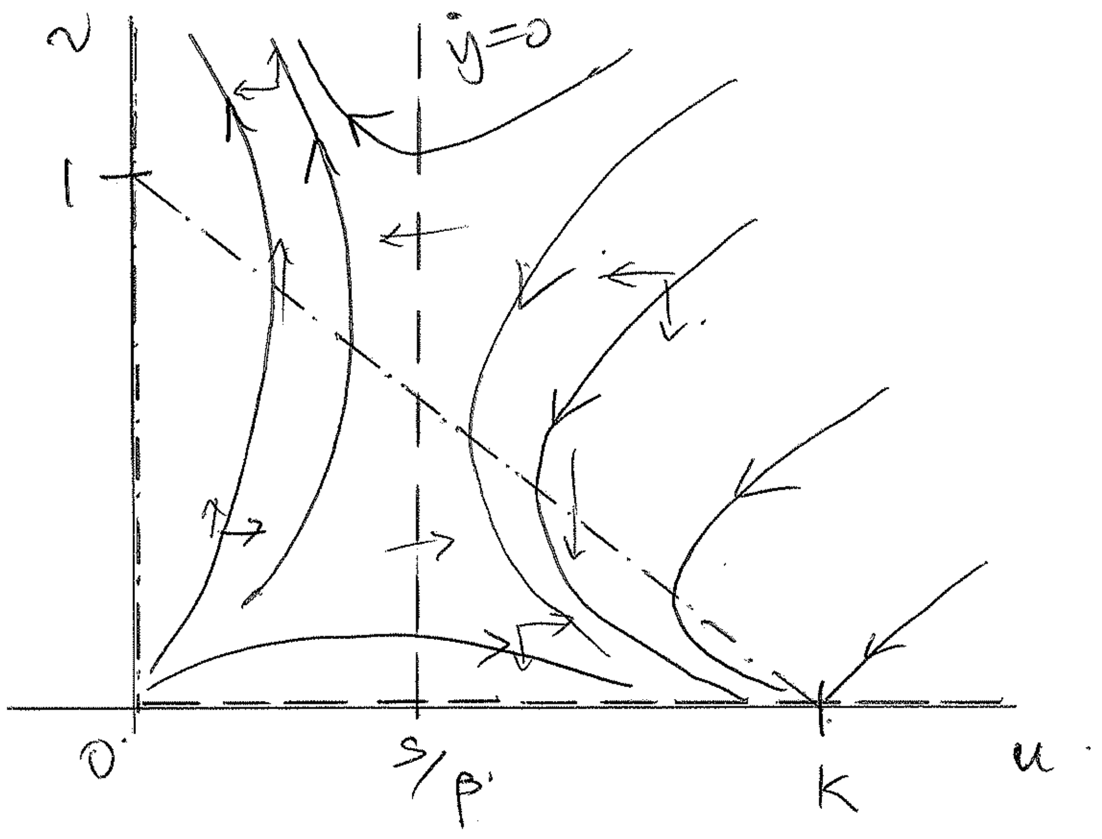
$\dot{u} = 0$ when $u = 0$ or $v = r\left(1 - \frac{u}{K}\right)$

$\dot{v} = 0$ when $v = 0$ or $u = s/\beta$.

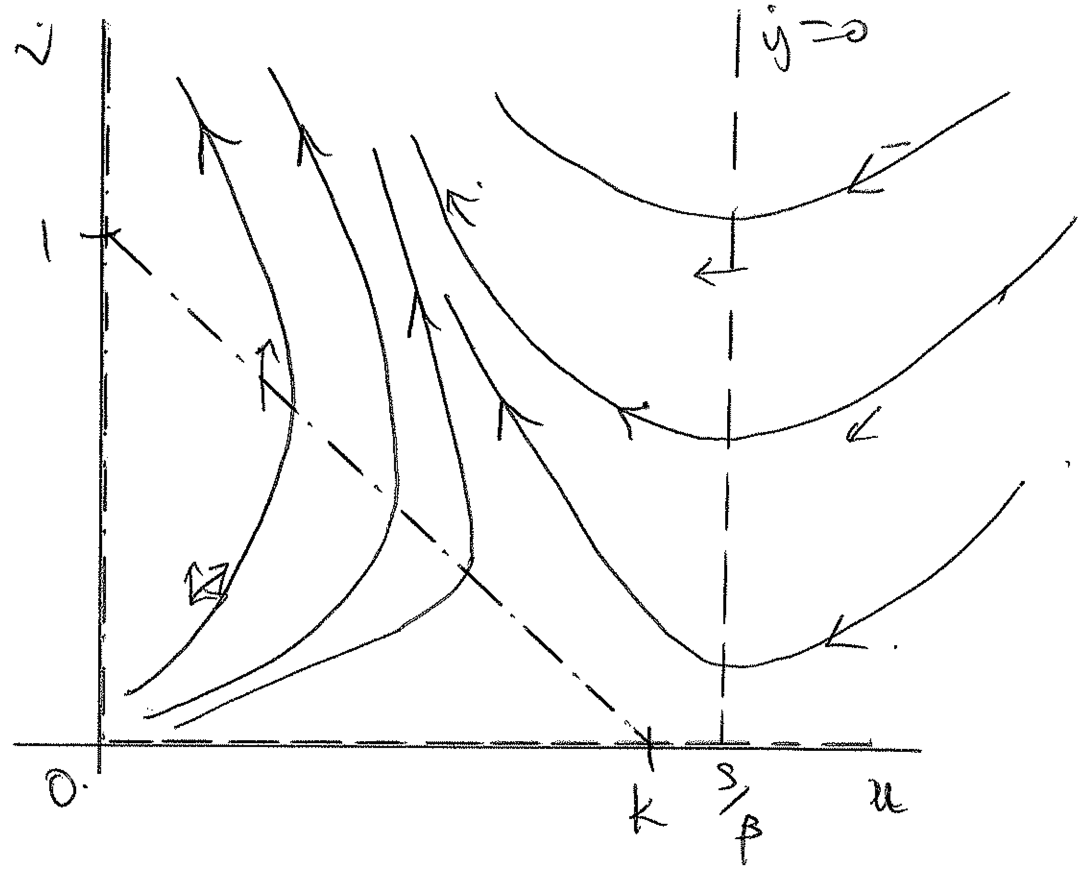
Flows $\dot{u} \gtrless 0$ when $v \lessgtr r(1 - u/K)$ $\rightleftarrows$

$\dot{v} \gtrless 0$ when $u \lessgtr s/\beta$ $\updownarrow$.

If $\beta K > s$ then $K > s/\beta$.



If $\beta K < s$ or $K < s/\beta$.



If $\beta K > s$ then either $u \to K$ and $v \to 0$ as $t \to \infty$ or $u \to 0$ and $v \to \infty$ as $t \to \infty$, depending on initial conditions.

If $\beta K < s$ then all solutions have $u \to 0$ and $v \to \infty$ as $t \to \infty$.

This is a reasonably good model for v low but as v becomes larger, the model predicts that v will increase indefinitely. Hence the model is not so good when v is large. In reality v cannot increase indefinitely as there will be other factors that limit the growth of this species

3. (i) Nullclines: $\dot{x}=0$ $\quad 3x^2+3y^2-6x=0.$

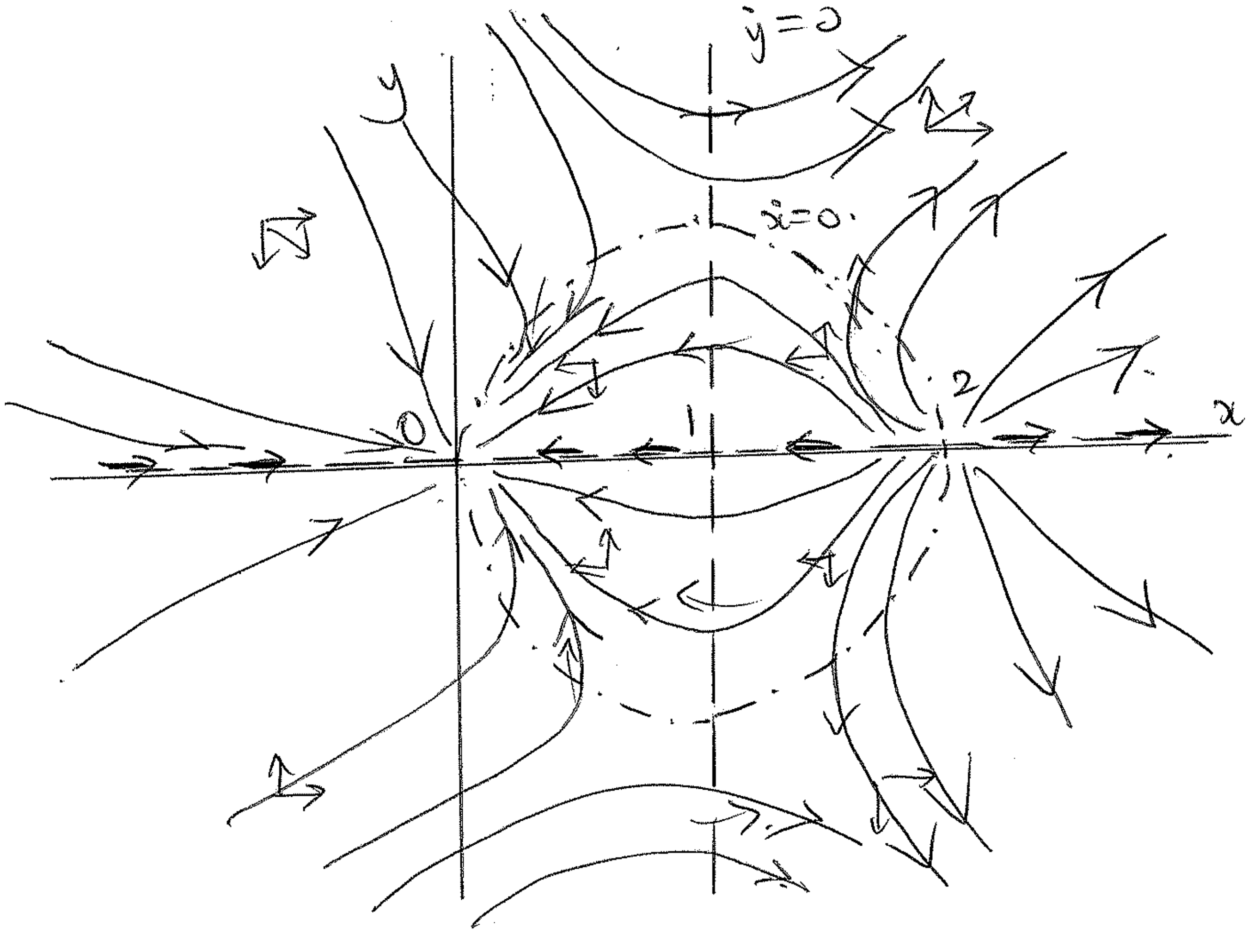
$$x^2-2x+1+y^2=1.$$
$$(x-1)^2+y^2=1.$$

$\dot{y}=0$ $\quad y=0$ or $x=1.$

Flows. $\dot{x}\gtrless 0$ if $3x^2+3y^2-6x\gtrless 0$
or $(x-1)^2+y^2\gtrless 1$

$\dot{y}>0$ if $y>0$ and $x>1$
or $y<0$ and $x<1.$

$\dot{y}<0$ if $y>0$ and $x<1.$
or $y<0$ and $x>1.$



(ii) If this system is a gradient system then

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}.$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(3x^2 + 3y^2 - 6x\right) = 6y.$$

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left(6xy - 6y\right) = 6y = \frac{\partial f}{\partial y}.$$

Hence the system is a gradient system.

$$-\frac{\partial F}{\partial x} = 3x^2 + 3y^2 - 6x.$$

$$\text{so} \quad F(x,y) = -x^3 - 3xy^2 + 3x^2 + h_1(y).$$

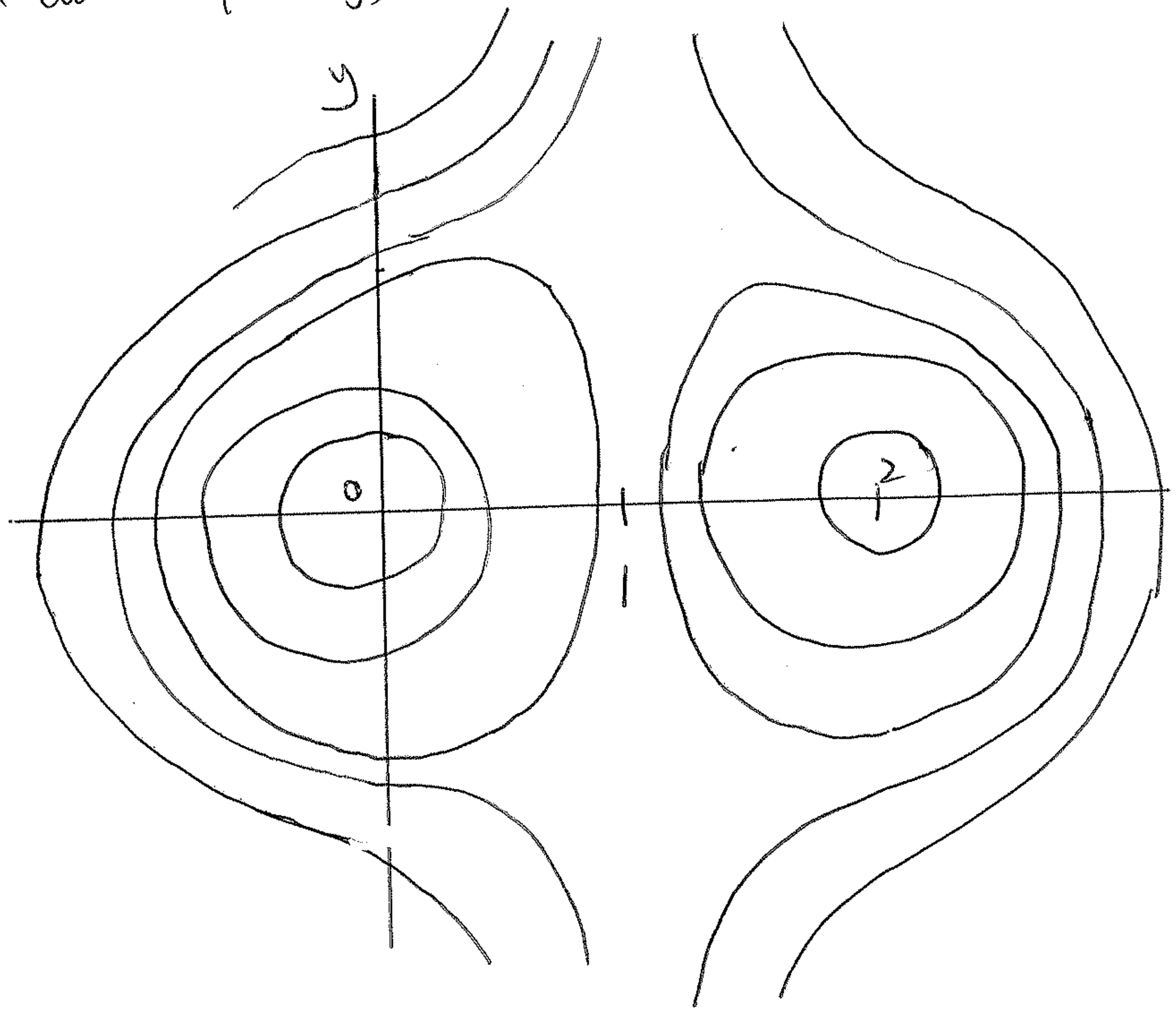
$$-\frac{\partial F}{\partial y} = 6xy - 6y$$

$$\text{so} \quad F = -3xy^2 + 3y^2 + h_2(x).$$

Comparing these we get

$$F(x,y) = -x^3 + 3y^2 + 3x^2 - 3xy^2.$$

(iii) Level curves of $F(x,y)$



4. Try $V = ax^2 + bxy + cy^2$.

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt}$$

$$= (2ax + by)(-x^3 + y^4) + (bx + 2cy)(-y^3 - 3xy^3)$$

$$= -2ax^4 + 2axy^4 - bx^3y + by^5$$
$$- bxy^3 - 3bx^2y^3 - 2cy^4 - 6cxy^4.$$

$$= -2ax^4 + (2a - 6c)xy^4 - bx^3y + by^5$$
$$- bxy^3 - 3bx^2y^3 - 2cy^4.$$

$$= -2ax^4 + (2a - 6c)xy^4 - 2cy^4 \quad \text{if } b = 0.$$

$$= -2ax^4 - 2cy^4 \quad \text{if } a = 3c.$$

$$< 0 \quad \text{if } a > 0, \; c > 0.$$

Choose $c = 1$ then $a = 3$ and $V(x, y) = 3x^2 + y^2$

Hence $\frac{dV}{dt} < 0$ for all nhds of $(0, 0)$.

$V > 0$ for all neighbourhoods of $(0, 0)$.

and V, V_x, V_y are all continuous

Hence V is a Liapunov function and $(0, 0)$ is asymptotically stable.