THE UNIVERSITY OF SYDNEY

FACULTIES OF ARTS, ECONOMICS, EDUCATION,
ENGINEERING AND SCIENCE

MATH3063

DIFFERENTIAL EQUATIONS & BIOMATHS (N)

Semester 1 Time Allowed: Two Hours

June 2007 Lecturer: A. Nelson

THIS EXAM CONSISTS OF 4 PAGES NUMBERED 1 TO 4.
THERE ARE 4 QUESTIONS, NUMBERED 1 TO 4.

All **4** *questions may be attempted.*
Reasons and working must be given.

Each question is marked out of 25. Part marks as shown.

The following notation is assumed,

$$\dot{x} = \frac{dx}{dt}, \qquad \dot{y} = \frac{dy}{dt}.$$

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1. The growth of a population is modelled by the differential equation

$$\frac{dx}{dt} = f(x) = rx(1 - x/M),$$

where x is population density as a function of time t, and $r > 0$ and $M > 0$ are constants.

(a) (*i*) Sketch the graph of $f(x)$.
Hence sketch phase line, and determine the equilibrium populations and their stability. [4 **marks**]

(*ii*) Write down the dimension of the constants r, M in terms of population density and time dimensions $[x]$ and $[t]$.
Give a population model theoretic interpretation of the constants r and M. [4 **marks**]

(b) Consider now harvesting the population by a constant effort strategy.
This modelled by the equation,

$$\frac{dx}{dt} = f(x) - Ex = rx(1 - x/M) - Ex,$$

where $E \geq 0$ is a measure of the effort.

(*i*) Determine how the equilibria of this *equation*, (including therefore any equilibria with $x < 0$), and their stability vary with E.
Sketch a bifurcation diagram for equilibria as function of E for this equation.
Indicate clearly the stable and unstable branches. [8 **marks**]

(*ii*) Determine in terms of r and M the maximal sustainable yield and the value of E which gives this yield.
Work out the return time T_R at the maximal sustainable yield. [4 **marks**]

(*iv*) Discuss how the model predicts the population, and the yield will vary if the effort is gradually increased from 0, giving time for the system to keep near stable equilibrium. [5 **marks**]

2. (a) Consider the linear system

$$\dot{x} = 2x - y, \quad \dot{y} = -x + 2y.$$

(*i*) Determine the eigenvalues and corresponding eigenvectors for coefficient matrix of this system. [4 **marks**]

(*ii*) Sketch the phase portrait of the system.
Include in your diagram the systems nullclines and normal modes. [6 **marks**]

(b) Consider the non-linear system

$$\dot{x} = y - x^5, \quad \dot{y} = -x - y^5.$$

(*i*) Show that this is not a gradient system. [3 **marks**]

(*ii*) Verify that $V(x, y) = x^2 + y^2$ is a strict Liapunov function for the equilibrium point $(0, 0)$ of this system. [4 **marks**]

(c) Determine the equilibrium points of the non-linear system

$$\dot{x} = y, \quad \dot{y} = x + x^2 - y.$$

Carry out a linear analysis at each equilibrium point. [8 **marks**]

3. (a) What is a limit cycle? Make sketches to illustrate a stable and an unstable limit cycle. [4 **marks**]

(b) Consider the Van der Pol system

$$\frac{dx}{dt} = y - \left(\frac{x^3}{3} - x\right), \qquad \frac{dy}{dt} = -x.$$

(*i*) Show that the equilibrium point $(0, 0)$ of the Van der Pol system is a source. [4 **marks**]

(*ii*) Sketch the nullclines of the system.
Indicate the direction of flow across the nullclines. [4 **marks**]

(*iii*) Make a separate sketch showing, and describe the construction of, a region which traps trajectories.
What is the significance for this system of the fact that $(0, 0)$ is a source? [8 **marks**]

(c) State the Bendixson-Dulac Theorem on the non-existence of limit cycles.

Apply the theorem with $B(x, y) = \dfrac{1}{xy}$ to show that no general population model

$$\dot{x} = xf(x, y), \quad \dot{y} = yg(x, y),$$

for which the per capita growth rates satisfy $f_x < 0$ and $g_y < 0$, can have a limit cycle in the positive quadrant. [5 **marks**]

4. The following system of differential equations models the phenomenon of mutualism, where two or more species confer a positive benefit on each other:

$$\frac{dx}{dt} = rx(1 - x/K) + axy, \quad \frac{dy}{dt} = sy(1 - y/L) + bxy.$$

In this model x and y are population densities as function of time t, and the model parameters a, b, K, L, r and s are positive constants.

This model can be put in dimensionless form

$$\frac{dX}{d\tau} = X(1 - X) + \alpha XY, \quad \frac{dY}{d\tau} = \lambda\big(Y(1 - Y) + \beta XY\big),$$

by setting $X = x/K,\ Y = y/L,\ \tau = rt,\ \lambda = r/s,\ \alpha = aL/r,\ \beta = bK/s.$

You are not required to derive this dimensionless form.

Answer part (a) below by refering to the equations in x and y.

(a) Explain briefly the assumptions of this model of mutualism. [4 **marks**]

For the rest of this question work with the dimensionless form in X and Y

(b) The dimensionless system always has three equilibrium states, $(0, 0)$, $(1, 0)$ and $(0, 1)$ corresponding to one or other or both species being absent.

Show that the system has has a fourth equilibrium state $(\bar{X}, \bar{Y})$ with $\bar{X} > 0$ and $\bar{Y} > 0$, if and only if $\alpha\beta < 1$. [3 **marks**]

(c) Do a linear analysis to determine the stability of all the equilibria with $X \geq 0$ and $Y \geq 0$. [7 **marks**]

(d) Sketch the phase portrait in the X, Y-plane of the dimensionless system, showing nullclines and flow directions, one sketch for the case there is, and one sketch for the case where there is not, an equilibrium point $(\bar{X}, \bar{Y})$ with $\bar{X} > 0$ and $\bar{Y} > 0$. [7 **marks**]

(e) What are the ecological predictions of this model? [4 **marks**]