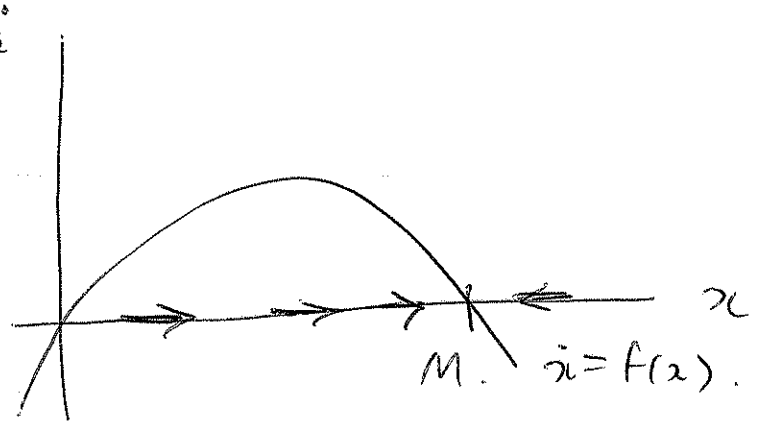
MATH 3063 – 2007 Exam – Solns + hints.

(a) (i)



Equilibrium populations are $x = 0$ (unstable) and $x = M$ (stable).

(ii) r has dimensions $[t]^{-1}$.
M has dimensions $[x]$

M is the carrying capacity of the environment for x.
r is the linear growth rate of the population.

(b). (i)
$$\frac{dx}{dt} = rx\left(1 - \frac{x}{M}\right) - Ex$$
$$= (r-E)x - \frac{rx^2}{M}.$$
$$= (r-E)x\left(1 - \frac{rx}{(r-E)M}\right).$$

Hence there are two steady states : $x = 0$ and $x = \dfrac{M(r-E)}{r}$.

Let $F = (r-E)x\left(1 - \frac{rx}{(r-E)M}\right)$.

$$\frac{dF}{dx} = (r-E) - \frac{2r}{M}x.$$

When $x=0$ $\quad F'(0) = r-E \begin{cases} > 0 & \text{if } r > E \\ < 0 & \text{if } E > r \end{cases}$

So $x = 0$ is $\begin{cases} \text{stable if } r < E \\ \text{unstable if } r > E \end{cases}$

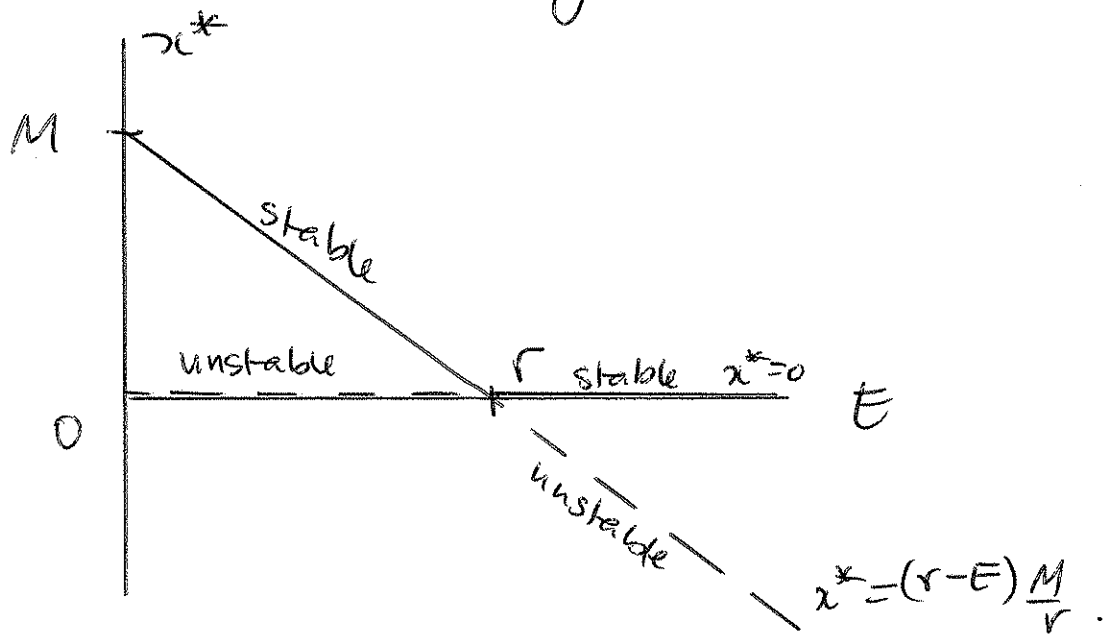
When $1 - \frac{rx}{(r-E)M} = 0$ $\quad x = \frac{r-E}{r}M$

$$F'\left(\frac{r-E}{r}M\right) = r-E-2(r-E)$$
$$= -(r-E)$$

So $F'\left(\frac{r-E}{r}M\right) \begin{cases} > 0 & \text{if } r < E \\ < 0 & \text{if } E < r \end{cases}$

So $x = \frac{r-E}{r}M$ is $\begin{cases} \text{stable if } r > E \\ \text{unstable if } r < E \end{cases}$

1(b)(i) ctd. Bifurcation diagram



(b)(ii) Maximum sustainable yield is max of Ex^*

$$Ex^* = \frac{M}{r}(r-E)E.$$

Maximum of this occurs when

$$\frac{d}{dE}\left(\frac{M}{r}(r-E)E\right) = 0$$

ie $\frac{M}{r}(r-E-E) = 0$

so $E = r/2$.

Hence Max. sustainable yield

$$= \frac{M}{r}\left(r-\frac{r}{2}\right)\frac{r}{2}$$

$$= \frac{Mr}{2}.$$

Return time to a stable equilibrium $x_i = x^*$ is

$$\frac{-1}{F'(x^*)} = \frac{-1}{F'\left(\frac{Mr}{2}\right)}$$

$$F'\left(\frac{Mr}{2}\right) = (r - E) - \frac{2r}{M}\frac{Mr}{2} \quad \text{at MSY.}$$

$$= \left(r - r/2\right) - \frac{2r^2}{2} \quad \text{substituting } E = r/2.$$

$$= \frac{r}{2} - r^2.$$

So return time is $\dfrac{1}{r(r - r/2)}$.

1(b)(iii). If the effort is slowly increased from zero then the equilibrium population will slowly decrease until it reaches 0 at $E = r$. When $E > r$ then the populations remain at $x^* = 0$.

As E increases from zero, the yield will increase until it reaches maximum sustainable yield at $E = r/2$. As E continues to increase, the yield decreases until it is zero at $E = r$. For $E > r$ the yield remains zero.

2(a) In matrix form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or} \quad \dot{\underset{\sim}{x}} = A \underset{\sim}{x}.$$

Characteristic eqn of A is $\lambda^2 - 4\lambda + 3 = 0$.

$(\lambda - 1)(\lambda - 3) = 0$ so $\lambda = 1, 3$.

Eigenvalues are $\lambda = 1$ and $\lambda = 3$.

Finding eigenvectors $\begin{pmatrix} u \\ v \end{pmatrix}$.

$$1\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$u = 2u - v$ $\qquad -u = -v$ or $u = v$.

$v = -u + 2v$

So eigenvector corresponding to $\lambda = 1$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$3\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

$3u = 2u - v$ $\qquad$ so $u = -v$.

$3v = -u + 2v$

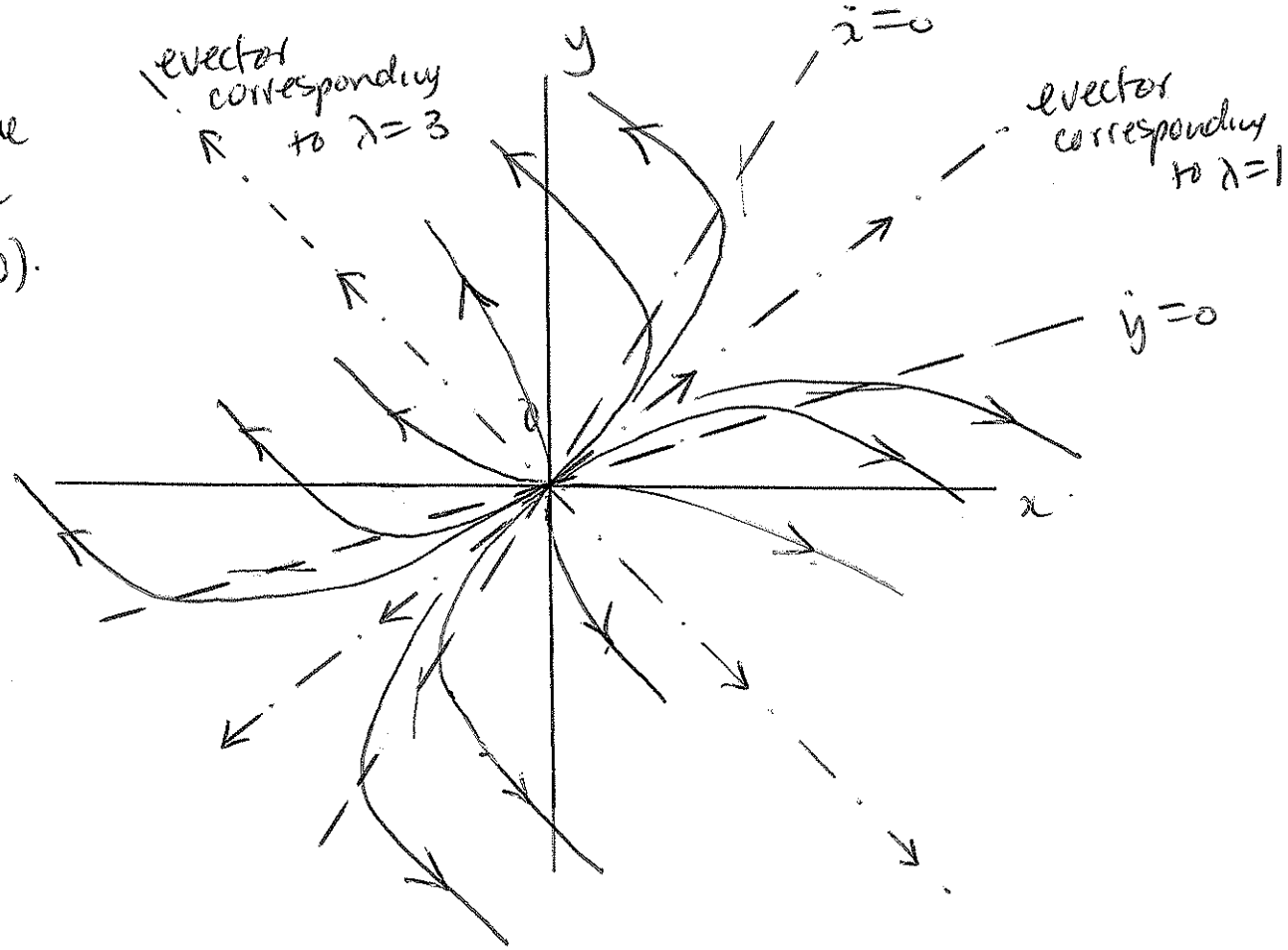
So eigenvector corresponding to $\lambda = 3$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(ii) Phase plane.

Nullclines $\dot{x} = 0$ $\quad y = 2x$

$\dot{y} = 0$ $\quad y = x/2$.

Unstable node at (0,0).



(b) (i) For a gradient system

$$\dot{x} = f(x,y) = -\frac{\partial F}{\partial x} \quad \text{and} \quad \dot{y} = g(x,y) = -\frac{\partial F}{\partial y}.$$

so $$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}.$$

Here $$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y - x^5) = 1.$$

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}(-x - y^5) = -1 \neq \frac{\partial f}{\partial y}$$

So this is not a gradient system.

2(b)(ii) $V(x,y)$ and both its first order partial derivatives are continuous

$V(x,y) = x^2 + y^2 > 0$ for all neighbourhoods of $(0,0)$.

$$\dot{V} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt}$$

$$= 2x(y - x^5) + 2y(-x - y^5)$$

$$= 2xy - 2x^6 - 2xy - 2y^6$$

$$= -2x^6 - 2y^6$$

< 0 in every neighbourhood of $(0,0)$.

So $V(x,y)$ is a strict Liapunov function for $(0,0)$.

(c). At equilibrium $\dot{x} = \dot{y} = 0$

so $y = 0$ and

$x + x^2 - y = 0$ so $x(1+x) = 0$, $x = 0, -1$.

Equilibria are $(0,0)$ and $(-1,0)$.

$$J = \begin{pmatrix} 0 & 1 \\ 1+2x & -1 \end{pmatrix}$$

At $(0,0)$

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

$\text{Det } J = -1 < 0$ so $(0,0)$ is a saddle.

At $(0,-1)$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

$\text{Det } J = 1 > 0$

$\text{Tr } J = -1 < 0$

$(\text{Tr } J)^2 - 4 \text{ Det } J = 1 - 4 = -3 < 0$

So $(0,-1)$ is a stable focus.

3(a). A limit cycle is an isolated periodic solution. By "isolated" we mean that there exists a region around the solution in the phase plane that contains no other periodic solution.

(b). (i). At equilibrium

$$y - (x^3/3 - x) = 0$$
$$-x = 0$$

So steady state is $x = 0$, $y = 0$.

$$J = \begin{pmatrix} -x^2+1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

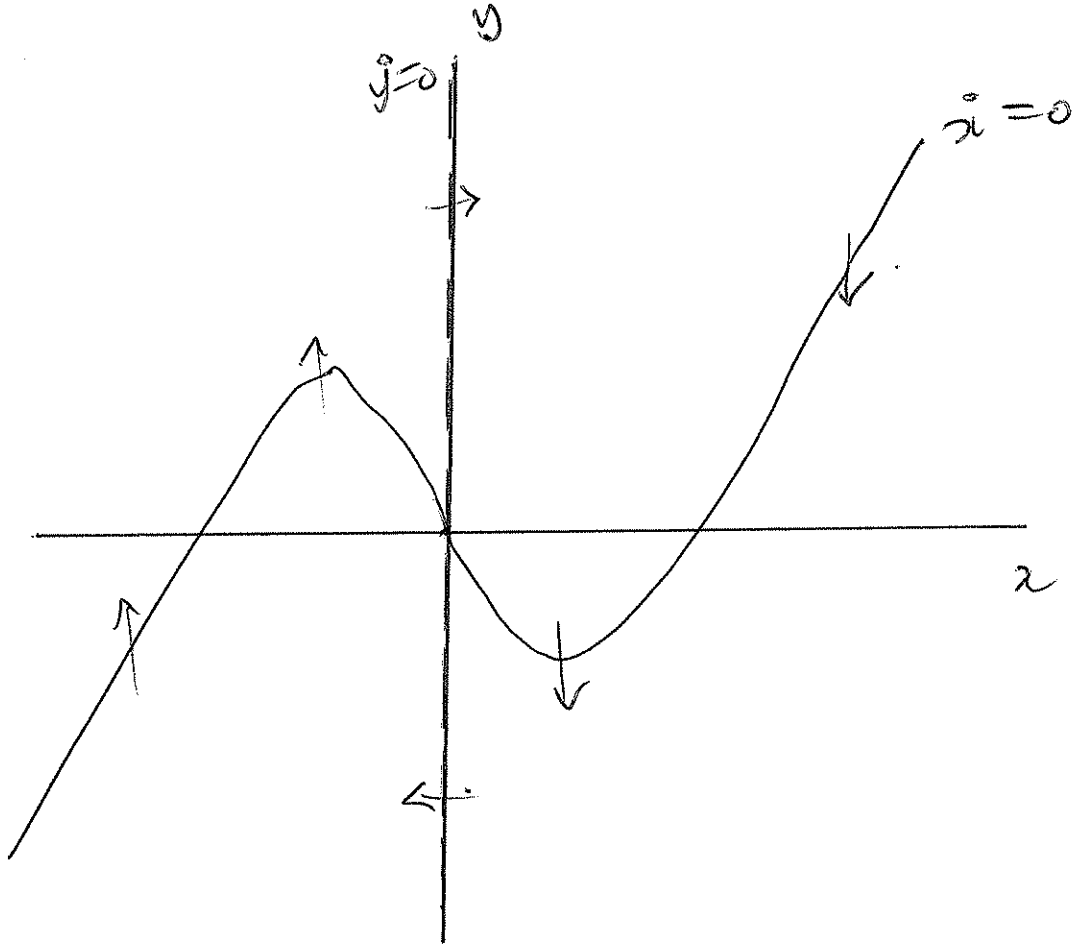
$$\left.\begin{aligned} \text{Det } J &= 1 > 0 \\ \text{Tr } J &= 1 > 0 \end{aligned}\right\} \text{ so } (0,0) \text{ is a source (unstable node or focus).}$$

(ii) Nullclines $\dot{x} = 0$ $\quad y = x^3/3 - x$.

$\dot{y} = 0$ $\quad x = 0$.

Flow $\dot{x} \gtrless 0$ if $y \gtrless x^3/3 - x$

$\dot{y} \gtrless 0$ if $x \lessgtr 0$



(c). The Bendixson-Dulac Theorem is not part of the course in 2009.

4. (a).

$$\frac{dx}{dt} = rx\left(1 - x/K\right) + axy$$

When $y=0$, then popn x grows logistically

When $y \neq 0$ then the growth of x increases with y at a rate proportional to the number of x and the number of y.

$$\frac{dy}{dt} = sy\left(1 - \frac{y}{L}\right) + bxy$$

When $x=0$ then y grows logistically

when $x \neq 0$ then the growth of y increases. The increased (additional) growth rate is proportional to x and proportional to y.

(b). At equilibrium

$$X\left((1-X) + \alpha Y\right) = 0.$$

$$\lambda Y\left((1-Y) + \beta X\right) = 0$$

so if $X \neq 0$ and $Y \neq 0$.

$$(1-X) + \alpha Y = 0 \qquad Y = \frac{1}{\alpha}(X-1).$$

$$(1-Y) + \beta X = 0$$

4(b) ctd. Sub in for Y into second eqn.

$$\left(1 - \tfrac{1}{\alpha}(X-1)\right) + \beta X = 0.$$

$$\left(1 + \tfrac{1}{\alpha}\right) + \left(\beta - \tfrac{1}{\alpha}\right) X = 0.$$

so provided $\frac{1}{\alpha} > \beta$, $\quad X = \dfrac{\left(1+\frac{1}{\alpha}\right)}{\frac{1}{\alpha} - \beta}$

$$Y = \frac{1}{\alpha}\left(\frac{1 + \frac{1}{\alpha}}{1/\alpha - \beta} - 1\right)$$

$$= \frac{1}{\alpha}\left(\frac{1 + \frac{1}{\alpha} - \left(\frac{1}{\alpha} - \beta\right)}{1/\alpha - \beta}\right)$$

$$= \frac{1}{\alpha}\left(\frac{1+\beta}{1/\alpha - \beta}\right)$$ again, provided $\frac{1}{\alpha} > \beta$.

or $\alpha\beta < 1$

(c). $$J = \begin{pmatrix} (1 - X + \alpha Y) - X & \alpha X \\ \lambda\beta Y & \lambda((1 - Y + \beta X) - Y) \end{pmatrix}.$$

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}.$$

Characteristic eqn in μ.

$$(1-\mu)(\lambda-\mu) = 0 \quad \text{so } \mu = 1, \lambda \text{ both +ve}$$

so $(0,0)$ is an unstable node

$$J(1,0) = \begin{pmatrix} -1 & 1 \\ 0 & +\beta\lambda \end{pmatrix}$$

Characteristic eqn (in μ)

$$(-1-\mu)(+\beta\lambda-\mu) = 0 \quad \text{so} \quad \mu = -1, +\beta\lambda$$

Different signs

So $(1,0)$ is a saddle

$$J(0,1) = \begin{pmatrix} +\alpha & 0 \\ \lambda\beta & -\lambda \end{pmatrix}$$

Characteristic eqn in μ.

$$(-\alpha-\mu)(-\lambda-\mu) = 0 \qquad \mu = +\alpha, -\lambda.$$

Different signs

So $(0,1)$ is a saddle

$$J\left(\frac{1+\frac{1}{\alpha}}{\frac{1}{\alpha}-\beta}, \frac{\frac{1}{\alpha}(1+\beta)}{1/\alpha-\beta}\right) = \begin{pmatrix} -\left(\frac{1+1/\alpha}{1/\alpha-\beta}\right) & \frac{1+\alpha}{1/\alpha-\beta} \\ \frac{\frac{\lambda\beta}{\alpha}(1+\beta)}{1/\alpha-\beta} & -\frac{\frac{\lambda}{\alpha}(1+\beta)}{1/\alpha-\beta} \end{pmatrix}$$

$$\text{or} \quad J(x^*, y^*) = \begin{pmatrix} -x^* & \alpha x^* \\ \lambda\beta y^* & -\lambda y^* \end{pmatrix}$$

$$\text{Det } J = \lambda x^* y^* - \lambda\alpha\beta x^* y^*$$

$$= \lambda x^* y^* (1-\alpha\beta) > 0 \text{ if } \alpha\beta < 1.$$

$$\text{Tr } J = -x^* - \lambda y^* < 0.$$

So (x^*, y^*) is either a stable node or stable focus.

$$(\text{Tr } J)^2 - 4\,\text{Det } J = (-x^* - \lambda y^*)^2 - 4\lambda x^* y^* (1-\alpha\beta)$$

$$= x^{*2} + 2\lambda x^* y^* + \lambda^2 y^{*2} - 4\lambda x^* y^* + 4\alpha\lambda\beta x^* y^*$$

$$= x^{*2} - 2\lambda y^* x^* + \lambda^2 y^{*2} + 4\alpha\lambda\beta x^* y^*$$

$$= (x^* - \lambda y^*)^2 + 4\alpha\beta\lambda x^* y^*.$$

$$> 0.$$

So (x^*, y^*) is a stable node.

(d). Nullclines $\dot{x} = 0$ $x = 0$ or $y = \frac{1}{\alpha}(x-1)$

$\dot{y} = 0$ $y = 0$ or $y = 1 + \beta x$.

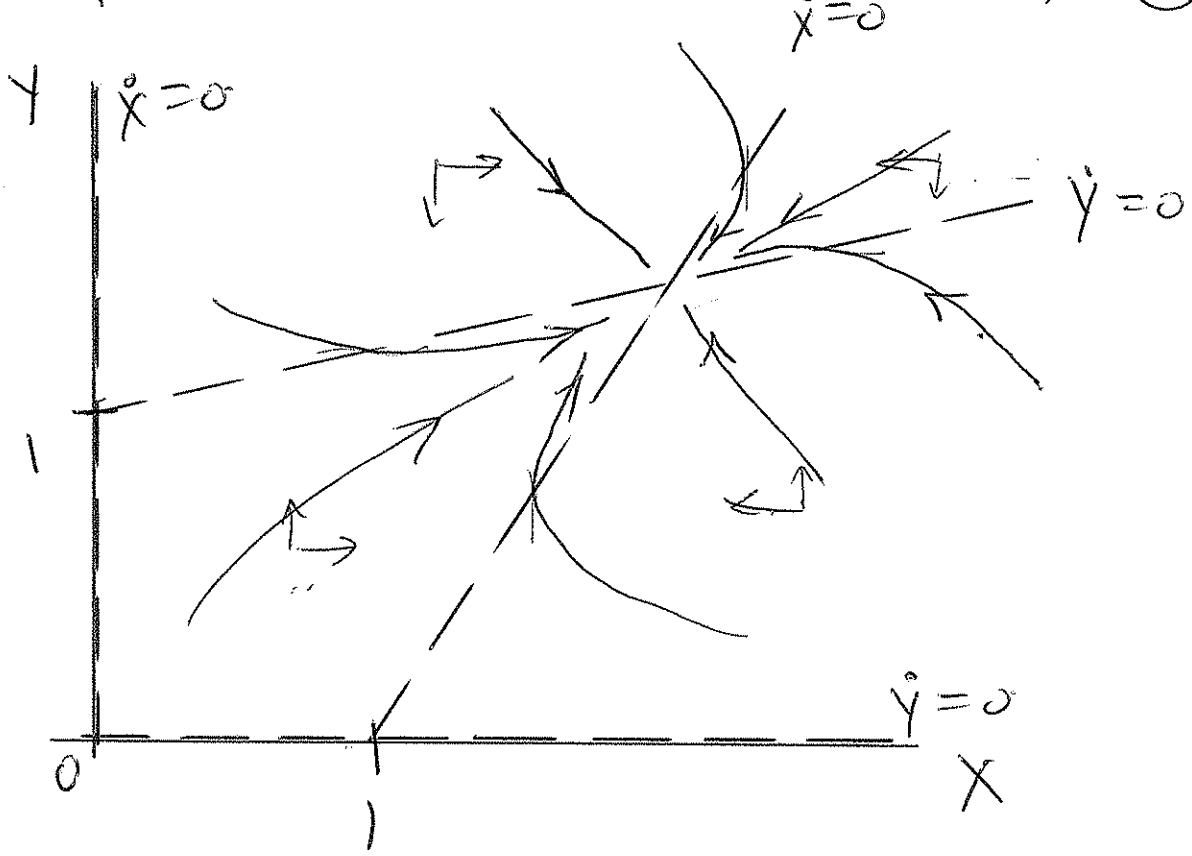
Flow $\dot{x} \gtrless 0$ if $y \gtrless \frac{1}{\alpha}(x-1)$

$\dot{y} \gtrless 0$ if $y \lessgtr 1 + \beta x$

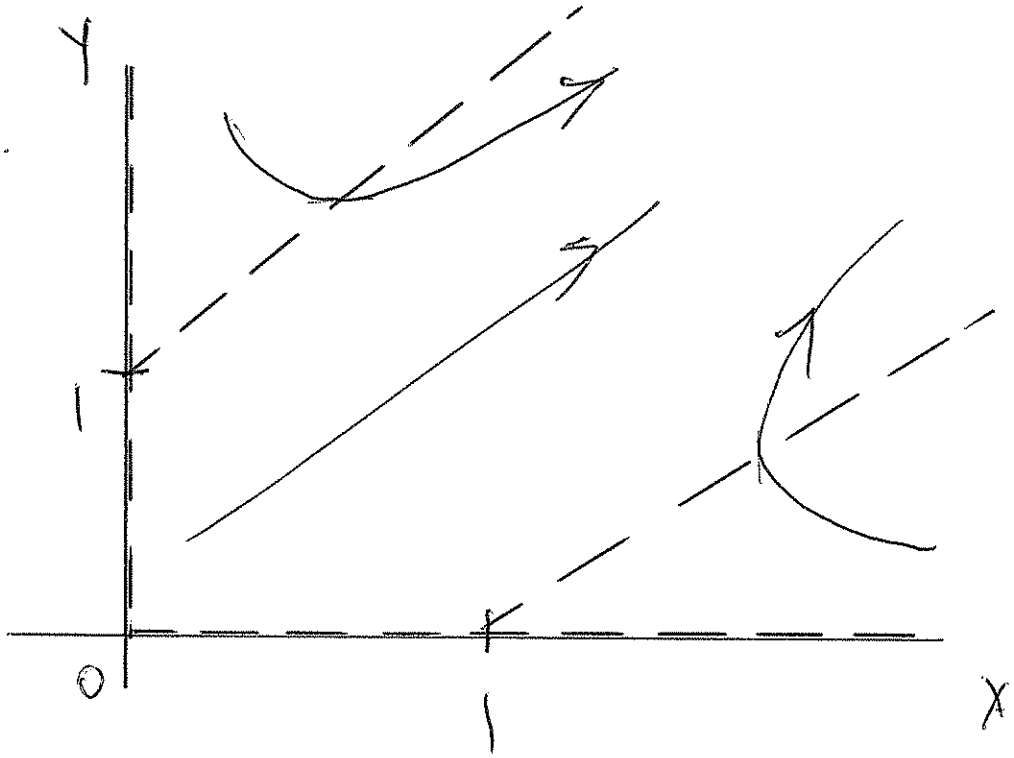
4(d) ctd.

$\alpha\beta < 1$.

With non-zero soln.



$\alpha\beta > 1$.



(e) If the effect of mutualism is smaller ($\alpha\beta < 1$) then the popns go to a steady state (X^*, Y^*) where $X^* > 1$ and $Y^* > 1$ so at a higher value than each popn would achieve on its own. If the effect of mutualism is stronger ($\alpha\beta > 1$) then both pops go to infinity as $t \to \infty$.