THE UNIVERSITY OF SYDNEY

Faculties of Arts, Economics, Education, Engineering and Science

MATH3063

Differential Equations & Biomaths (N)

Semester 1 Time Allowed: Two Hours

June 2010 Lecturer: M.R. Myerscough

This exam consists of 6 pages numbered 1 to 6.
There are 4 questions, numbered 1 to 4.

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*All **4** questions may be attempted.*
Reasons and working must be given.

Each question is marked out of 20. Part marks as shown.

Calculators are not permitted.

The following notation is assumed,

$$\dot{x} = \frac{dx}{dt}, \qquad \dot{y} = \frac{dy}{dt}.$$

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1. (a) An insect population, density x, is modelled by the equation

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{M}\right)\left(1 + \frac{x}{N}\right) = f(x)$$

where r, M and N are positive constants.

(*i*) Show that there are two steady state populations. [2 **marks**]

(*ii*) Find the stability of the steady state populations using linear stability analysis. [3 **marks**]

(*iii*) Sketch the phase portrait of the equation. [2 **marks**]

(*iv*) The insects are sprayed frequently with insecticide which reduces the population by a constant amount h per unit time. Draw a bifucation diagram showing how steady state populations x^* change as h changes. Indicate the stable and unstable branches. [2 **marks**]

(*v*) Can the insect population be exterminated by this spraying regime? Why or why not? [2 **marks**]

(b) Find a suitable Liapunov function of the form $V(x,y) = ax^2 + cy^2$ to show that $(0,0)$ is a stable steady state of

$$\begin{aligned}\frac{dx}{dt} &= -x - xy^3\\ \frac{dy}{dt} &= -y + y^2x^2.\end{aligned}$$

[5 **marks**]

(c) What is meant by an *unstable limit cycle*? Use a sketch to illustrate your answer. Include in your sketch any steady states that are associated with the limit cycle. [3 **marks**]

2. Rabbits, which are vigorous, rapidly reproducing, invasive herbivores, are introduced into an area of natural bushland where they come into competition with a species of native herbivore. A model is proposed for this process:

$$\begin{aligned}\frac{du}{dt} &= u(1-u-\alpha v)\\ \frac{dv}{dt} &= \rho v(1-\beta u)\end{aligned}$$

where u is the number of native herbivores and v is the number of rabbits.

(a) Explain the assumptions of the model. Why is it a suitable model for this particular situation? [2 **marks**]

(b) Find the steady states of the model, noting any restrictions on parameters that are needed for the steady state to be positive. [2 **marks**]

(c) Find the linear stability of these steady states. [5 **marks**]

(d) Sketch the phase plane for the model. There are two possible qualitatively different configurations of the phase plane. You should sketch both, each on a separate diagram. [5 **marks**]

(e) What happens to the populations of rabbits and native herbivores as $t \to \infty$? [2 **marks**]

(f) The Department of the Environment hires a contractor who undertakes to shoot a certain number of rabbits each week in the bushland. The equations for the interaction are modified to take account of this. They then become

$$\begin{aligned}\frac{du}{dt} &= u(1-u-\alpha v)\\ \frac{dv}{dt} &= \rho v(1-\beta u)-h\end{aligned}$$

where h is the rate at which rabbits are removed. Sketch the phase plane for the case when $h/\rho < 1/\alpha$. Will the contractor's work eliminate the rabbits? Explain your answer. [4 **marks**]

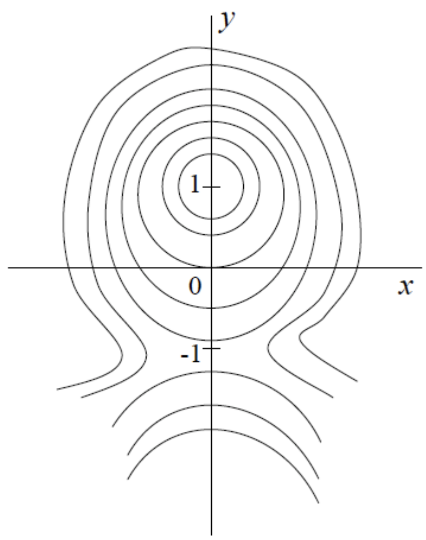
3. (a) In this part of the question on the Lotka-Volterra equations, you do not have to show working but you should answer from your knowledge of the Lotka-Volterra model and its results.

   (*i*) Write down the Lotka-Volterra equations for two interacting populations, one a predator, population P, and the other its prey, population N. [2 **marks**]

   (*ii*) What do these equation predict about the behaviour of these populations? [1 **mark**]

   (*iii*) How do the populations change when both predator and prey are harvested with constant effort harvesting? [1 **mark**]

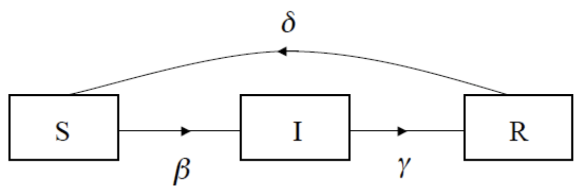
   (b) The function $W(x, y) = x^2 + y^3 - 3y$ has a local minimum at $(0, 1)$ and a saddle at $(0, -1)$. The level curves of $W(x, y)$ are illustrated below.



   (*i*) Write down the set of first order Hamiltonian ODE's for which $W(x, y)$ is the Hamiltonian. Draw the phase plane for this system and indicate the steady states and their nature. You do *not* need to indicate the direction of flow in this case. [3 **marks**]

   (*ii*) Write down the first order gradient system of two ODE's for which $W(x, y)$ is the gradient function. Draw the phase plane for this system and indicate the steady states and their nature. You *do* need to indicate the direction of flow in this case. [4 **marks**]

(c) In an SIRS epidemic, members of the removed class slowly lose immunity and return to the susceptible class. The dynamics of this type of epidemic are given in the diagram below:



The equations that model this scenario are:

$$\begin{aligned}\frac{dS}{dt} &= -\beta SI + \delta R\\ \frac{dI}{dt} &= \beta SI - \gamma I\\ \frac{dR}{dt} &= \gamma I - \delta R\end{aligned}$$

where β, γ and δ are all positive parameters.

(i) Assume that the total number of individuals in the population is constant; that is, $N = S + I + R$ where N is constant. Hence show that the model can be rewritten using only two equations:

$$\begin{aligned}\frac{dS}{dt} &= f(S, I)\\ \frac{dI}{dt} &= g(S, I).\end{aligned}$$

(You should specify explicitly the expressions for $f(S, I)$ and $g(S, I)$.) [2 **marks**]

(ii) Show that at most two steady states exist. Write down the condition for the steady state with $S \neq 0$ and $I \neq 0$ to exist. [3 **marks**]

(iii) Show that, if two steady states exist, then the disease will always be present in the community. [4 **marks**]

4. (a) State the Poincaré-Bendixson Theorem [3 **marks**]

(b) Consider the equations

$$\begin{aligned}\frac{dx}{dt} &= -x^3 + \mu x - y\\ \frac{dy}{dt} &= \epsilon x\end{aligned}$$

where $0 < \epsilon \ll 1$ and μ is a parameter which can vary.

(i) Show that these equations have one steady state [1 **mark**]

(ii) Linearise the equation in the neighbourhood of the steady state and find the eigenvalues of the Jacobian matrix J in terms of the paramenter μ. [4 **marks**]

(iii) Show that the system undergoes a Hopf bifurcation when $\mu = 0$. [3 **marks**]

(iv) What is the stability of the steady state when $\mu < 0$? What is the stability when $\mu > 0$? Assuming that the steady state is asymptotically stable when the Hopf bifurcation occurs, for what values of μ will a limit cycle exist? [3 **marks**]

(v) Draw the phase plane when the limit cycle exists. You should show the nullclines, the flow and the limit cycle on your diagram. What effect does the small paramenter ϵ have on the phase flow? [6 **marks**]

(vi) Using the information in your phase plane, sketch a graph of x and y as a function of t for the limit cycle solution. [3 **marks**]