THE UNIVERSITY OF SYDNEY

Faculties of Arts, Economics, Education, Engineering and Science

MATH3063

Differential Equations & Biomaths (N)

Semester 1 Time Allowed: Two Hours

June 2011 Lecturer: M.R. Myerscough

This exam consists of 6 pages numbered 1 to 6.
There are 4 questions, numbered 1 to 4.

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*All **4** questions may be attempted.*
Reasons and working must be given.

Each question is marked out of 20. Part marks as shown.

Calculators are not permitted.

The following notation is assumed,

$$\dot{x} = \frac{dx}{dt}, \qquad \dot{y} = \frac{dy}{dt}.$$

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1. (a) A mouse population at a small rural industrial site is modelled by

$$\frac{dx}{dt} = rx\left(\frac{x}{P} - 1\right)\left(1 - \frac{x}{K}\right) = f(x)$$

where r, P and K are positive constants and $P \ll K$.

(*i*) Show that there are three steady state populations. [2 **marks**]

(*ii*) Find the stability of the steady state populations using linear stability analysis. [4 **marks**]

(*iii*) Sketch the phase portrait of the equation. [2 **marks**]

(*iv*) The mouse population is subject to a control program using mouse traps. The equation for x then becomes

$$\frac{dx}{dt} = f(x) - Tx.$$

Sketch $f(x)$ and Tx on the same axis for a range of values of T. Mark in the steady states for one value of T and indicate their stability. [3 **marks**]

(*v*) Draw a bifurcation diagram showing how steady state populations x^* change as T is changed. Hence, or otherwise, explain what will be the outcome of a program of intensive trapping. [3 **marks**]

(*vi*) During periods of good rainfall, K (which is effectively the carrying capacity for the population) can increase dramatically while P remains unchanged. Draw a bifurcation diagram for the model with trapping showing how the steady state populations x^* change as K increases when T is held constant. [2 **marks**]

(b) What is meant by a *saddle node bifurcation*? Use sketches of a bifurcation diagram and some phase planes to illustrate your answer. In your sketch label all important features. [4 **marks**]

2. Two strains of bacteria, X and Y, whose densities are x and y respectively, interact when they are grown together in culture. The interaction is modelled by

$$\frac{dx}{dt} = x(1 - x - ay)$$

$$\frac{dy}{dt} = y(1 - y(b + x))$$

where a and b are positive constants with $a \neq b$ and x and y are both greater than or equal to 0.

(a) Is the interaction between X and Y a predator-prey interaction, competition or a mutualistic interaction? [1 **mark**]

(b) Show that there are three steady states with either $x = 0$, $y = 0$ or both. [2 **marks**]

(c) Find the stability of all steady states which have $x = 0$ or $y = 0$ or both. [4 **marks**]

(d) Show that there exists at most two steady states with both $x > 0$ and $y > 0$. [2 **marks**]

(e) Sketch the phase plane in the following cases:

(i) There is no steady state with both x and y non-zero.

(ii) There is one steady state with both x and y non-zero. (You should consider only the case where $b > a$.)

(iii) There are two steady states with both x and y non-zero.

[8 **marks**]

(f) Describe each of the possible outcomes of the interaction between X and Y as $t \to \infty$. [3 **marks**]

3. (a) Explain the difference between linear stability and asymptotic stability. You may use diagrams to illustrate your explanation if you wish to. [2 **marks**]

(b) Consider the following set of autonomous ordinary differential equations

$$\begin{aligned} \frac{dx}{dt} &= x(\alpha - x) - y \\ \frac{dy}{dt} &= x - 1. \end{aligned}$$

(*i*) Show that these equations have exactly one steady state. [1 **mark**]

(*ii*) Show that a Hopf bifurcation occurs when $\alpha = 2$. What is the stability of the steady state when $\alpha < 2$ and when $\alpha > 2$? [4 **marks**]

(*iii*) Draw the phase plane when the limit cycle exists. You may assume that the steady state is asymptotically stable when the Hopf bifurcation occurs. Show the limit cycle clearly on your diagram. [4 **marks**]

(c) Consider the system of equations

$$\begin{aligned} \frac{dx}{dt} &= 2y \\ \frac{dy}{dt} &= 1 - x^2. \end{aligned}$$

(*i*) Show that this system is Hamiltonian. Find the Hamiltonian function $H(x, y)$. [3 **marks**]

(*ii*) Find where $H(x, y)$ has critical points and find the nature of these critical points. Hence draw the phase plane for the system. You do not need to show the direction of the flow, only the shape of the trajectories. What type of steady state is $(1, 0)$? [6 **marks**]

4. (a) Show that $G(N, P) = -c \ln N - a \ln P + dN + bP$ where a, b, c and d are all positive parameters and N and P are non-negative, is a first integral of the Lotka-Volterra equations,

$$\begin{aligned} \frac{dN}{dt} &= aN - bNP \\ \frac{dP}{dt} &= -cP + dNP. \end{aligned}$$

[4 **marks**]

(b) Consider the following model for excitation in an excitable medium:

$$\begin{aligned} \frac{dx}{dt} &= -x(x-1)(x-2) - y \\ \frac{dy}{dt} &= \epsilon(x - \beta). \end{aligned}$$

Here x represents the level of excitation, y is the recovery variable, $\beta \geq 0$ is a parameter of the system, and $0 < \epsilon \ll 1$.

(i) Sketch the phase plane for $\beta = 0$. Using this phase plane diagram, explain what is meant by *threshold behaviour* in excitable media. You may label your diagram and refer back to it in your explanation. [5 **marks**]

(ii) For what values of β do relaxation oscillations occur? Sketch a phase plane when β is in this range. Draw the periodic orbit and indicate the fast and slow parts of the oscillation on the phase plane. [6 **marks**]

(c) Using the Liapunov function $V(x, y) = x^2 + y^2$, show that all solutions of

$$\begin{aligned} \frac{dx}{dt} &= x(y-1) \\ \frac{dy}{dt} &= y(x-1) \end{aligned}$$

that start in the domain $x^2 + y^2 < 1$ tend asymptotically to the steady state $(0, 0)$.

[5 **marks**]