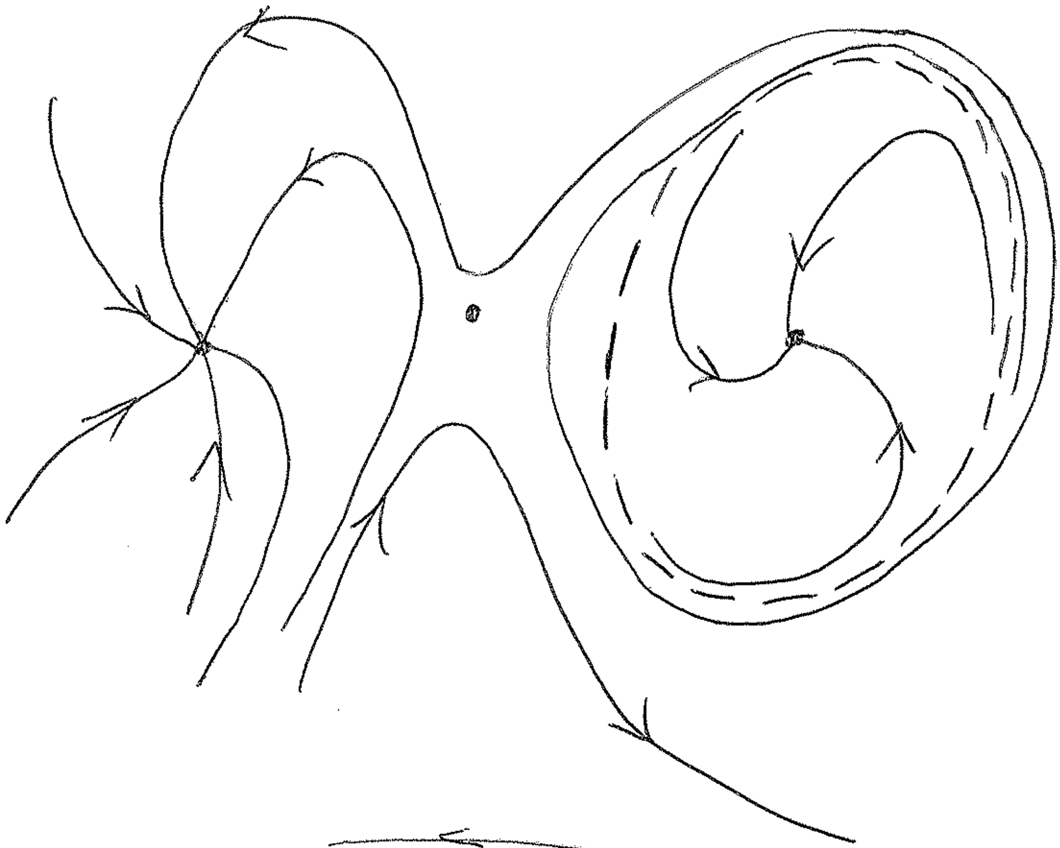
MATH 3063 - Tutorial Week 11.

1 (i).



ii

Phase planes which meet the criteria of the questions can be drawn in many ways (particularly (i)). These are just two examples

2(i). Let $x = r\cos\theta$ $\quad y = r\sin\theta$.

$$\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - \frac{d\theta}{dt} r\sin\theta.$$

$$\frac{dy}{dt} = \frac{dr}{dt}\sin\theta + \frac{d\theta}{dt} r\cos\theta.$$

$$x^2 + y^2 = r^2.$$

Sub into eqns

① $\frac{dr}{dt}\cos\theta - \frac{d\theta}{dt} r\sin\theta = -r\sin\theta + r\cos\theta(3-4r+r^2).$

② $\frac{dr}{dt}\sin\theta + \frac{d\theta}{dt} r\cos\theta = r\cos\theta + r\sin\theta(3-4r+r^2)$

① $\times\cos\theta$ + ② $\sin\theta$

$$\frac{dr}{dt}\cos^2\theta - \frac{d\theta}{dt} r\sin\theta\cos\theta + \frac{dr}{dt}\sin^2\theta + \frac{d\theta}{dt} r\cos\theta\sin\theta$$
$$= -r\cos\theta\sin\theta + r\cos^2\theta(3-4r+r^2) + r\sin\theta\cos\theta + r\sin^2\theta(3-4r+r^2).$$

$$\frac{dr}{dt}(\cos^2\theta + \sin^2\theta) = r(\cos^2\theta + \sin^2\theta)(3-4r+r^2).$$

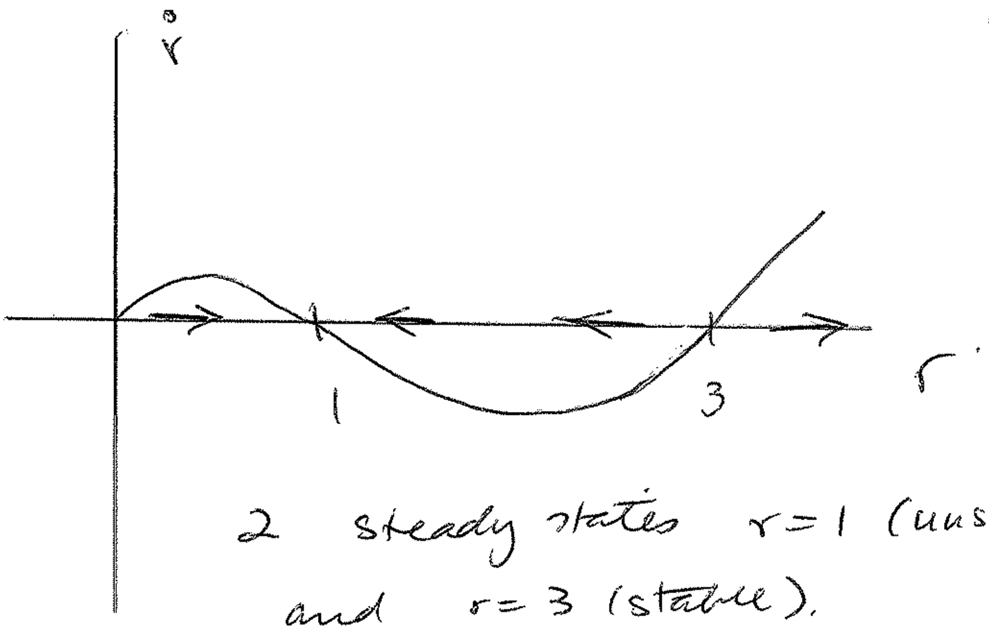
$$\frac{dr}{dt} = r(3-4r+r^2)$$
$$= r(1-r)(3-r).$$

① $\times -\sin\theta$ + ② $\times\cos\theta$

$$\frac{d\theta}{dt}\cancel{r}(\cos^2\theta + \sin^2\theta) = r(\sin^2\theta + \cos^2\theta)$$

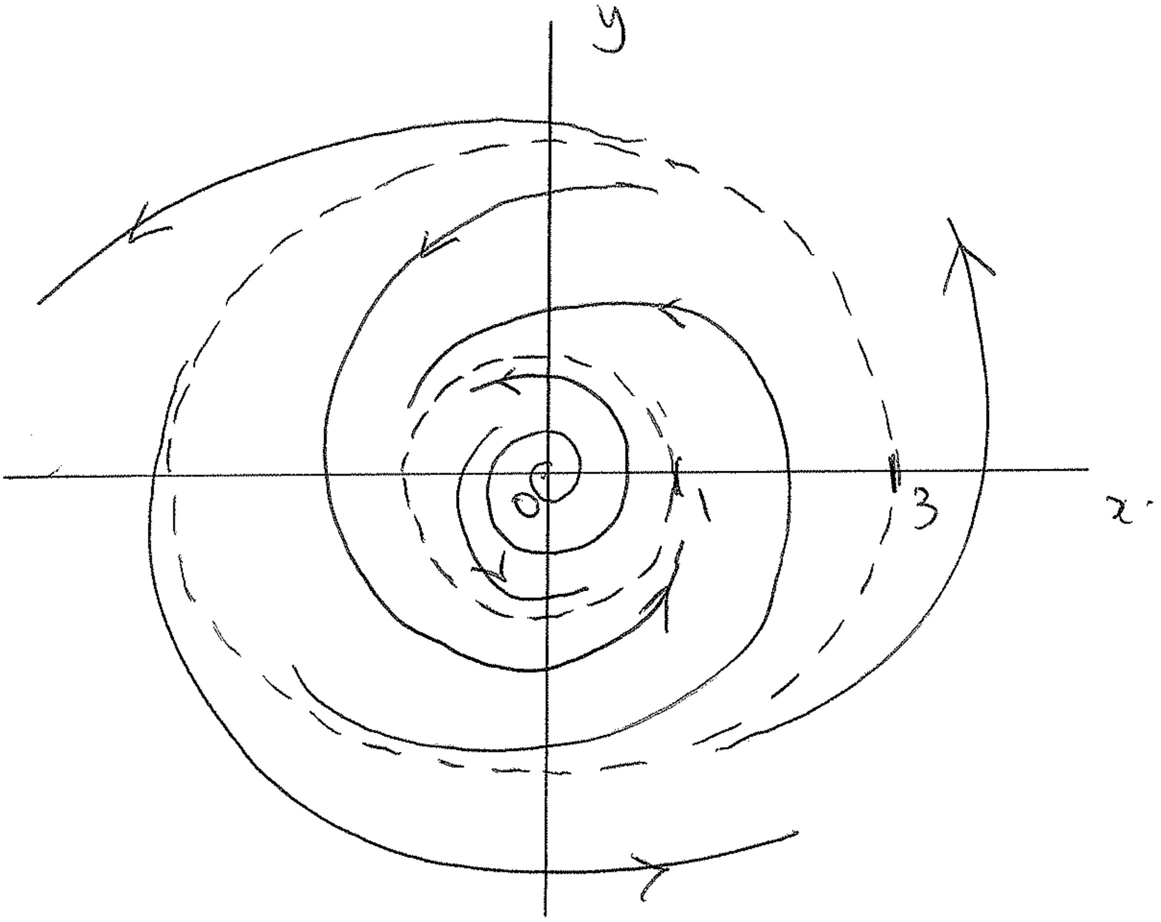
$$\text{so } \frac{d\theta}{dt} = 1.$$

2(ii).



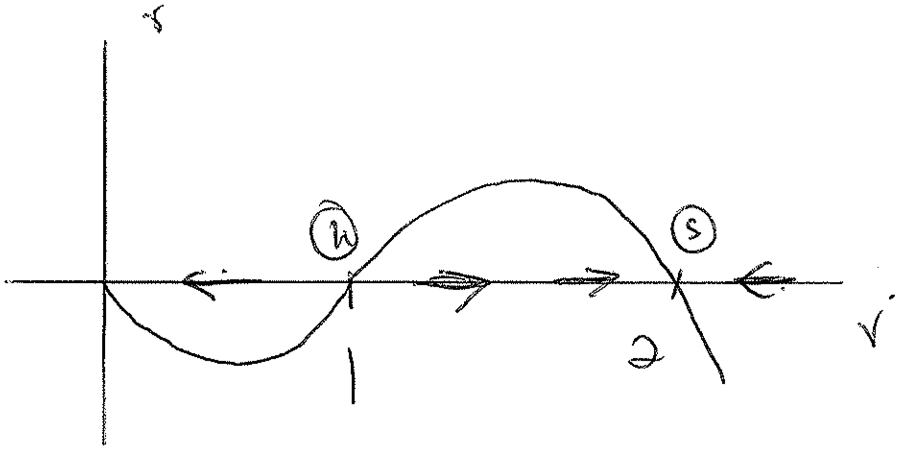
2 steady states $r=1$ (unstable) and $r=3$ (stable).

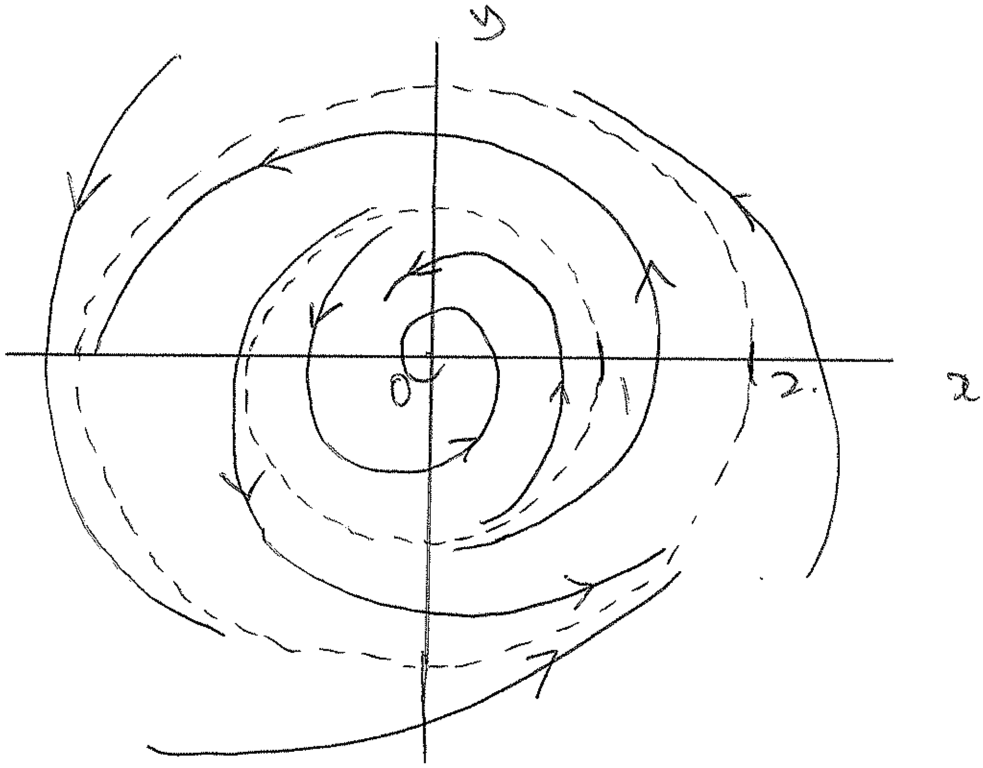
2(iii)



There are two limit cycles, a stable limit cycle at $r=1$ and an unstable limit cycle at $r=3$. The origin is an unstable focus

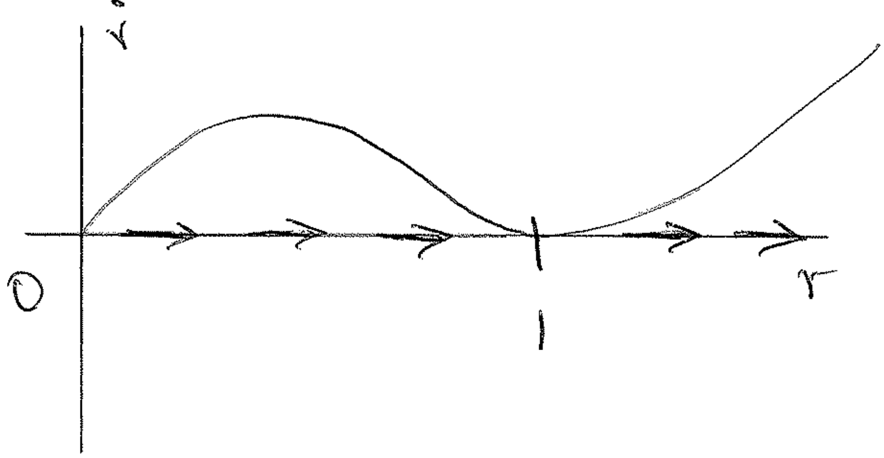
3 (i)



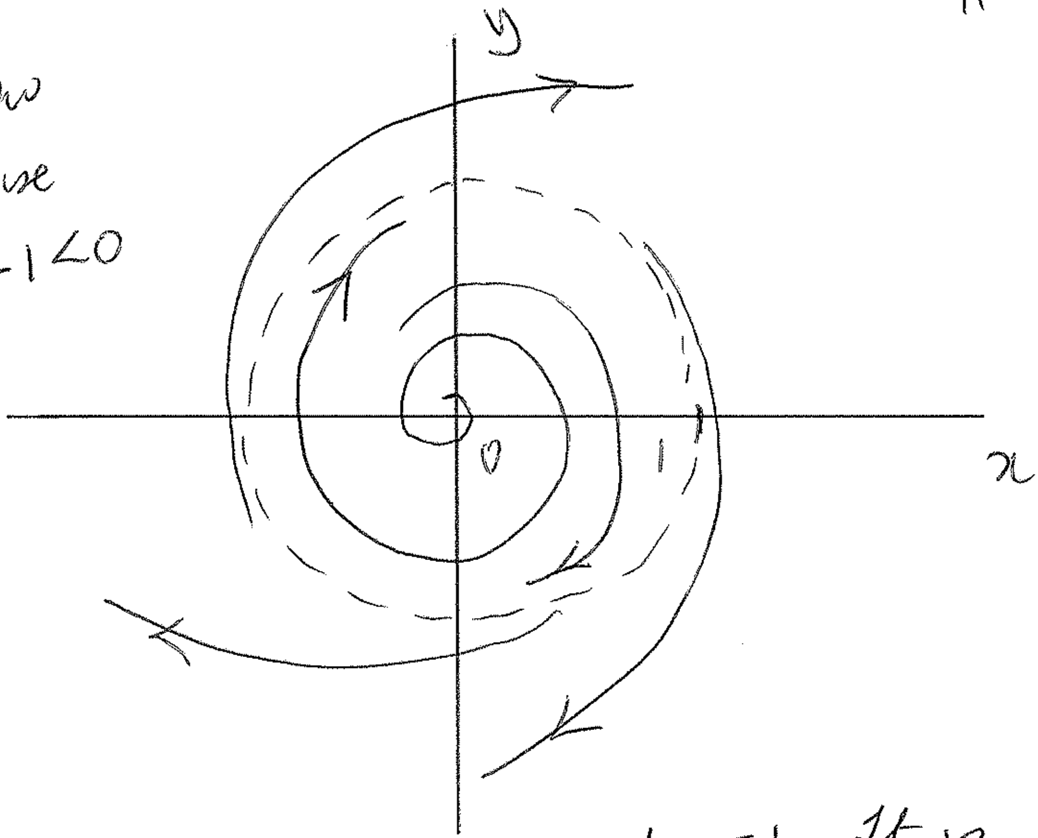


Two limit cycles - a stable limit cycle at $r=2$ and an unstable limit cycle at $r=1$. The origin is a stable focus.

3 (ii)

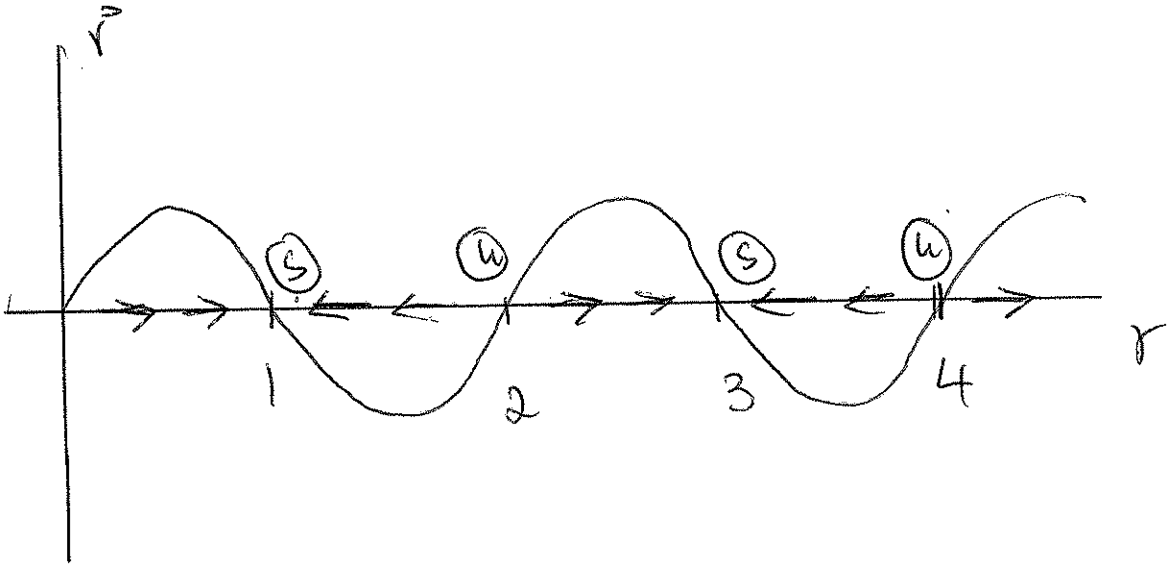


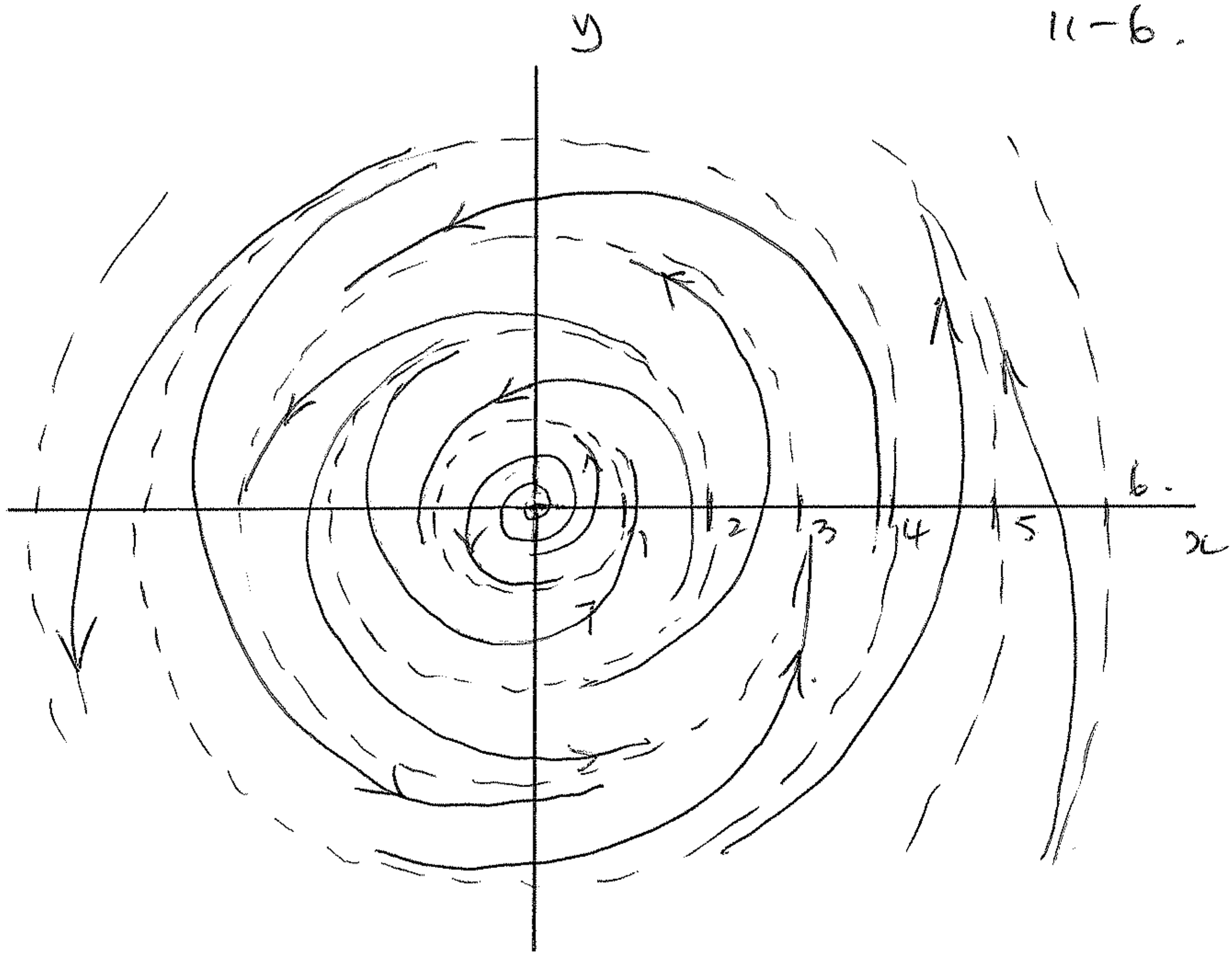
Note flow goes clockwise as $\dot{\theta} = -1 < 0$



One semi stable limit cycle at $r=1$. It is stable from the inside and unstable from the outside. The origin is an unstable focus.

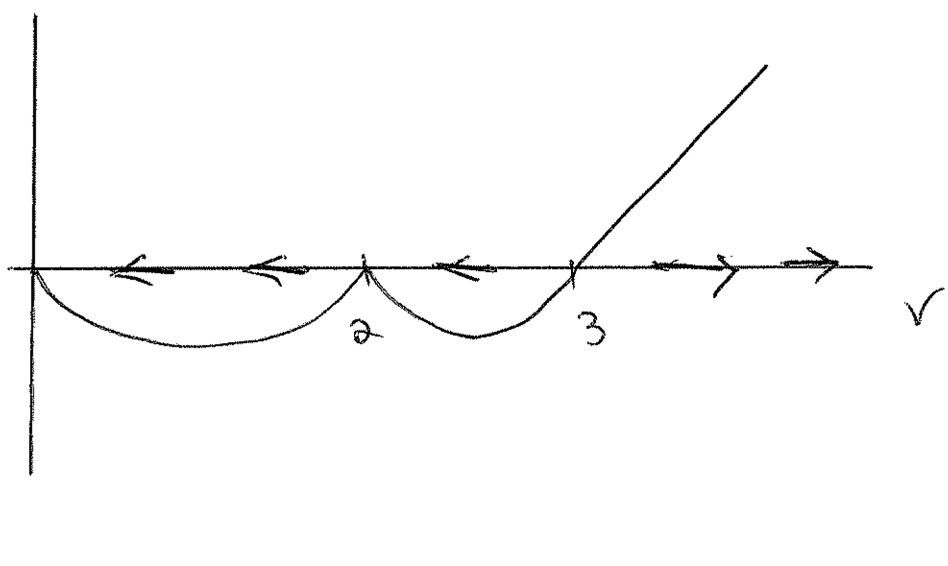
3 (iii)



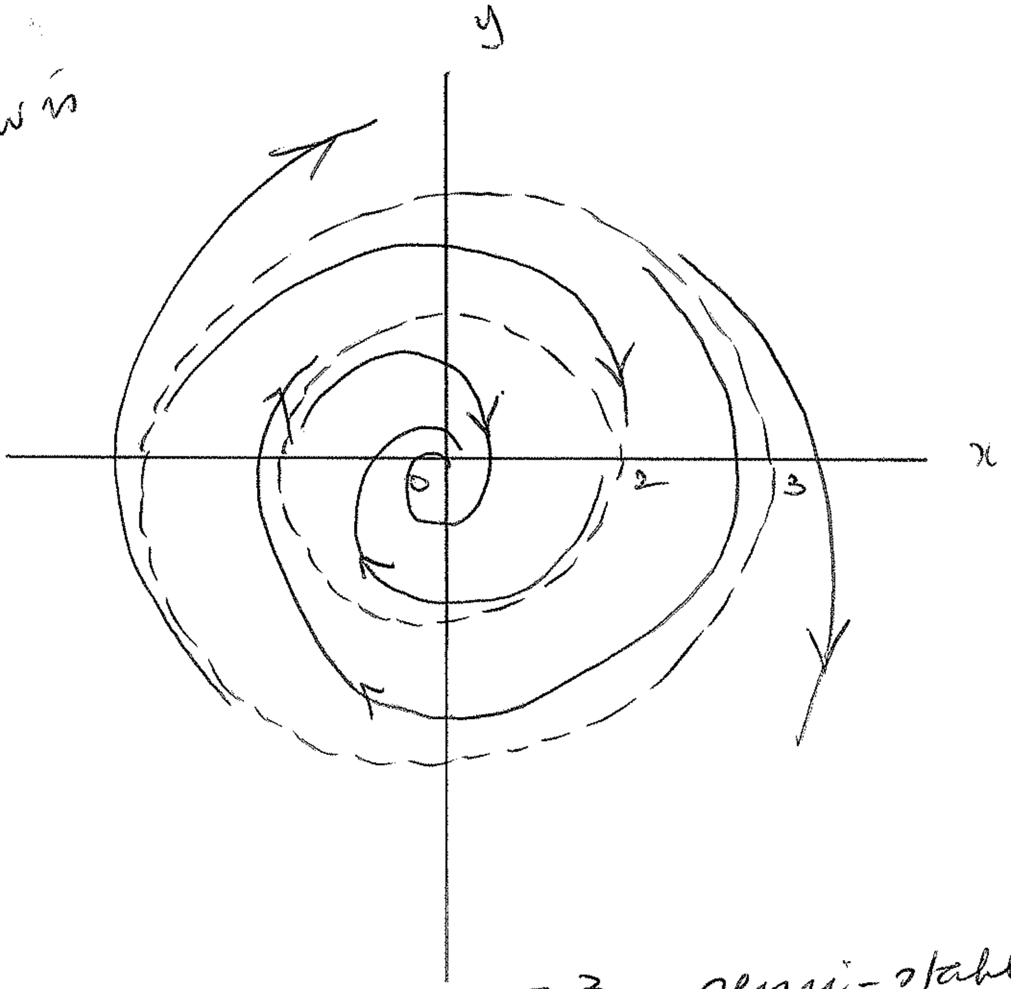


There are an infinite number of limit cycles. Those at $r = 2n-1$ $n \in \mathbb{N}$ are stable (odd numbers); those at $2r = 2n$ $n \in \mathbb{N}$ are unstable (even numbers). The origin is an unstable focus.

3 (iv).

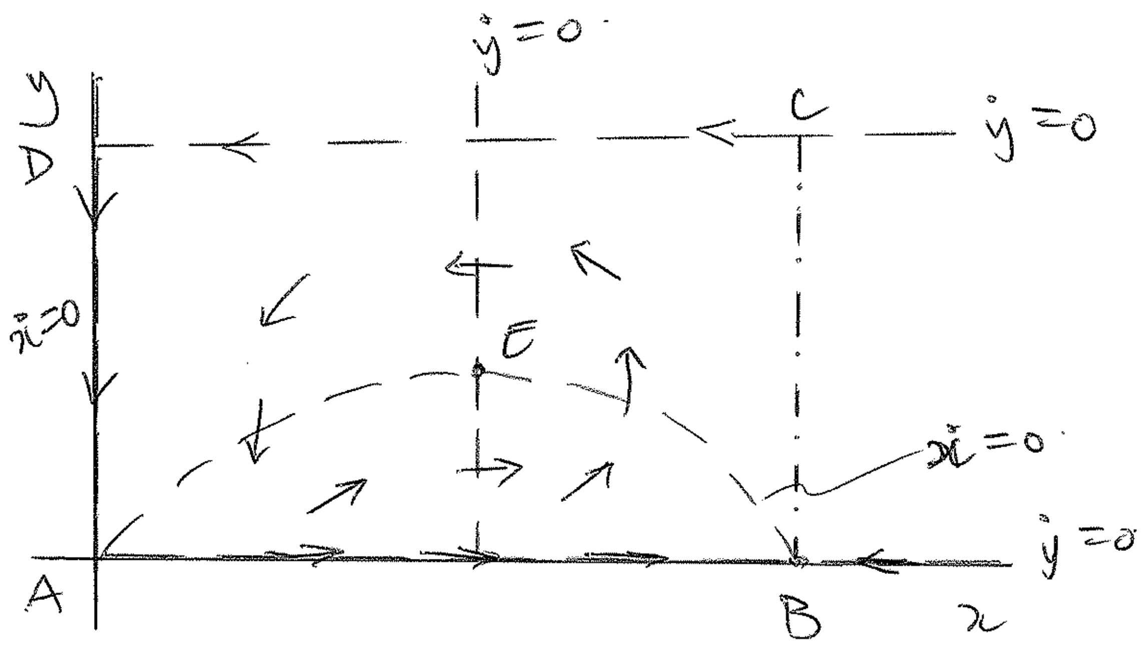


Note flow is clockwise as $\frac{d\theta}{dt} = -1$.



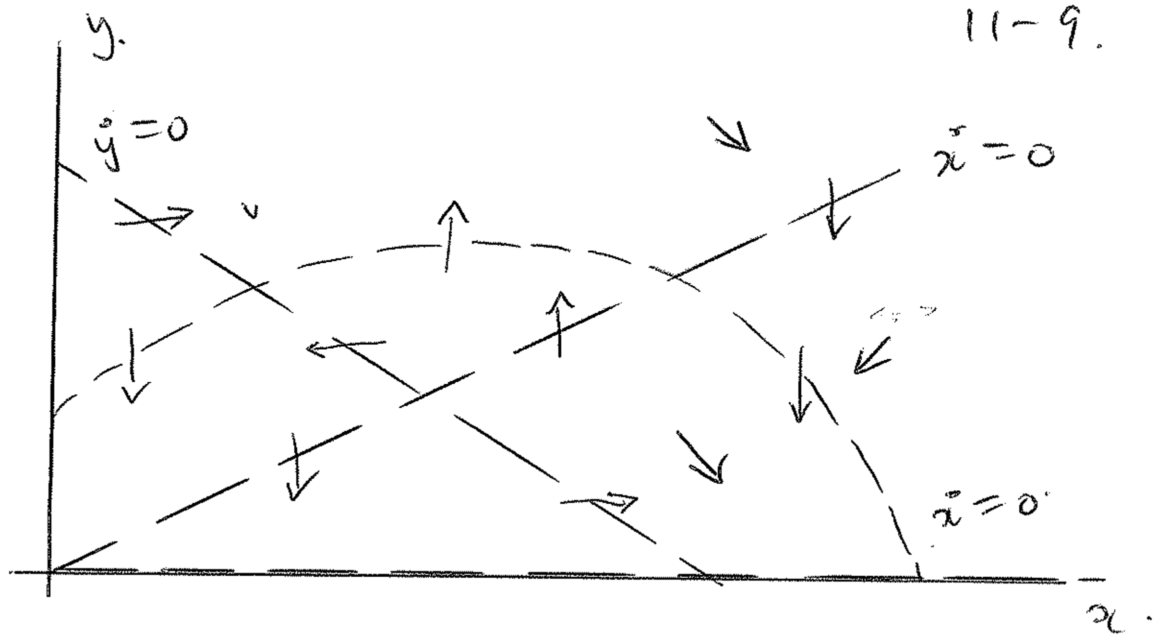
Unstable limit cycle at $r = 3$, semi-stable limit cycle at $r = 2$ (stable from the outside, unstable from the inside). The origin is a stable focus.

4(a)



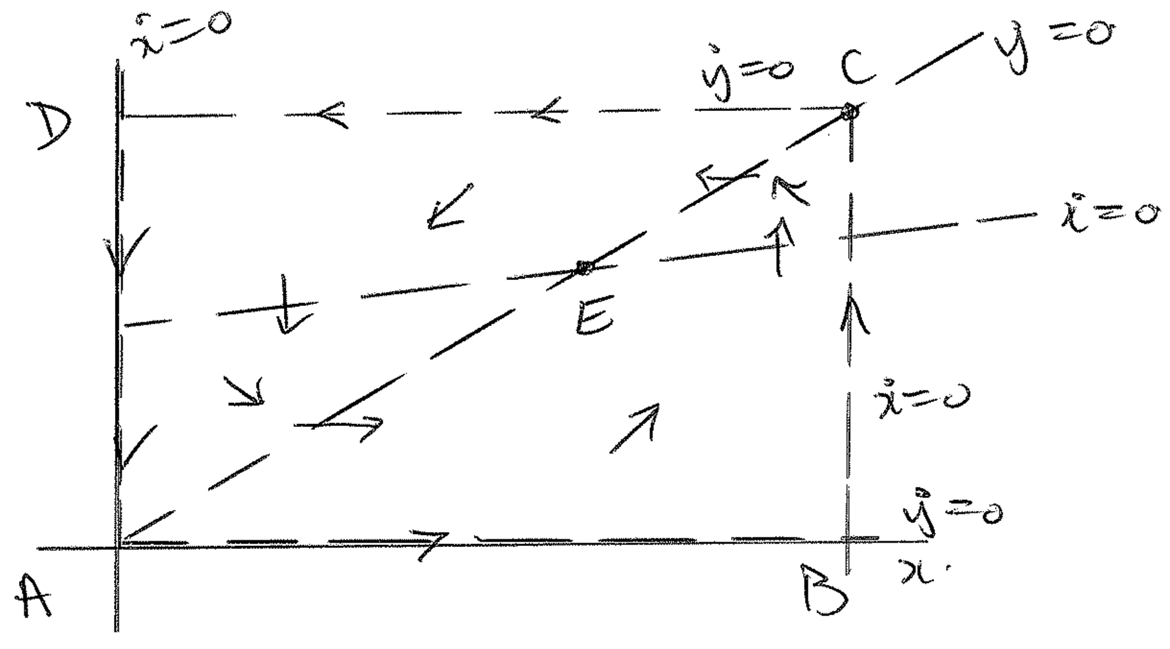
Label points A, B, C and D as shown. Flow cannot cross lines AD, AB and CD as these are isoclines with the flow running along them. On the line BC all horizontal flow is inwards. So a bounded set which no flow can leave. The steady state E is ~~an~~ unstable so by the Poincaré - Bendixson theorem there is a stable limit cycle inside ABCD.

4(b).



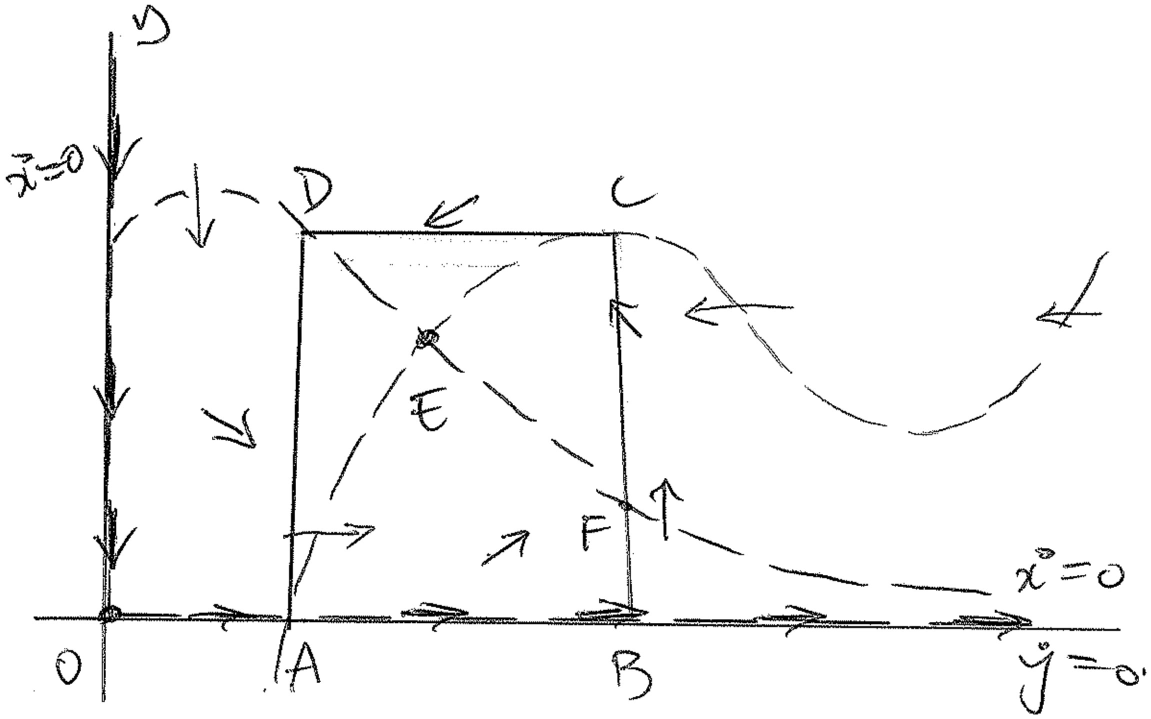
This phase plane is inconsistent (Try drawing the flows!) So the Poincaré Bendixson Theorem cannot be applied.

4(c).



All the lines AB, BC, CD and AD are isoclines with the flow running along them. Hence no flow can leave ABCD. Since E is unstable, by the Poincaré-Bendixson theorem there is a stable limit cycle inside ABCD.

4(d).



In this case it is hard to find a bounded set D s.t. no flow leaves D. The set ABCD as drawn allows flow to leave the set along the line BF.

We may still be able to construct a set D which no flow leaves by using curves line and arranging carefully that all flows go in D but we cannot apply the Poincaré - Bendixson theorem using a set D with straight-line boundaries.