MATH 3063 – Tutorial Week 12:

1. (i) At steady state

$$1-3x+ax^2y=0$$
$$2x-ax^2y=0$$

Add these equations to get

$$1-x=0 \quad \text{so } x=1.$$

$$2-ay=0 \quad \text{so } y=2/a.$$

There is only one steady state $(1, 2/a)$.

(ii) $J=\begin{pmatrix} -3+2axy & ax^2 \\ 2-2axy & -ax^2 \end{pmatrix}$.

when $x=1$, $y=2/a$

$$J=\begin{pmatrix} 1 & a \\ -2 & -a \end{pmatrix}.$$

Characteristic eqn $\lambda^2-(1-a)\lambda+a=0$.

So $\lambda=\dfrac{(1-a)\pm\sqrt{(1-a)^2-4a}}{2}$.

$Re(\lambda)=0$ when $\frac{1-a}{2}=0$ or $a=1$.

$\frac{d}{da}(Re\,\lambda)=-\frac{1}{2}\neq 0$ for any value of a.

So, a Hopf bifurcation occurs when $a=1$.

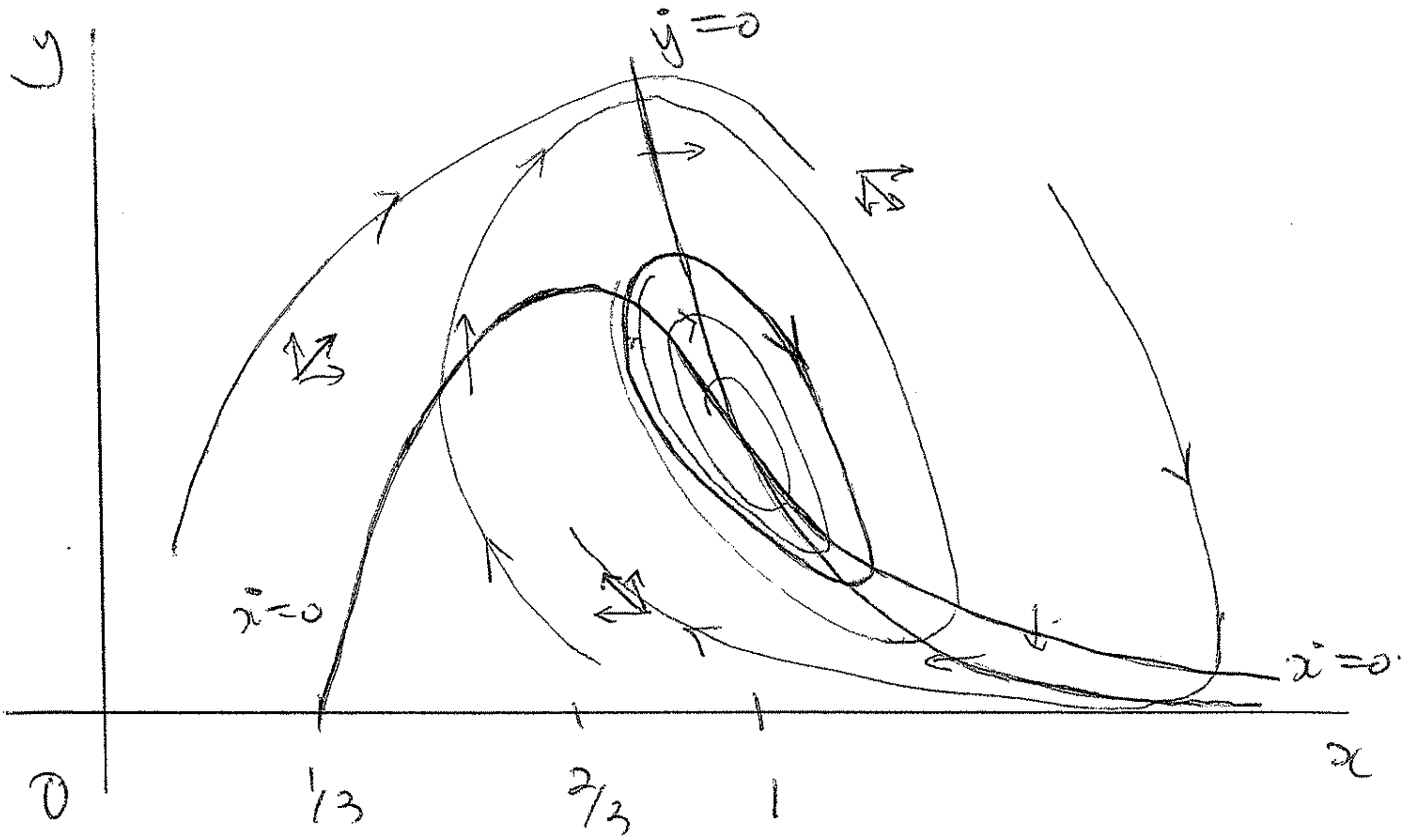
(iii) Nullclines $\dot{x}=0$ $y = \frac{3x-1}{ax^2}$

$\dot{y}=0$ $x=0$ or $y = \frac{2}{ax}$

Flow $\dot{x} \gtrless 0$ if $y \gtrless \frac{3x-1}{ax^2}$

$\dot{y} \gtrless 0$ if $y \lessgtr \frac{2}{ax}$ (note $x>0$, $a>0$)

When the stable limit cycle exists the focus at $(1, 2/a)$ is unstable so $Re(\lambda)>0$ so $a<1$.



2. (i) Set $\frac{dx}{dt} = y$ then eqn becomes

$$\frac{dy}{dt} - \mu(1-x^2)y + x = 0$$

or $$\frac{dy}{dt} = \mu(1-x^2)y - x.$$

(ii) At steady state $y = 0$ so

$$\mu(1-x^2)\cancel{y} - x = -x = 0.$$

The only steady state is $(0, 0)$.

$$J = \begin{pmatrix} 0 & 1 \\ 2\mu xy - 1 & x^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}$$

Characteristic eqn $\lambda^2 - \mu\lambda + 1 = 0$.

$$\lambda = \frac{+\mu \pm \sqrt{\mu^2 - 4}}{2}.$$

$\mathrm{Re}(\lambda) = \mu/2 = 0$ when $\mu = 0$.

$\frac{d}{d\mu}(\mathrm{Re}(\lambda)) = 1/2 \neq 0$ for all μ.

Hence, a Hopf bifurcation occurs at $\mu = 0$.

(iii) Nullclines $\dot{x} = 0$ $y = 0$

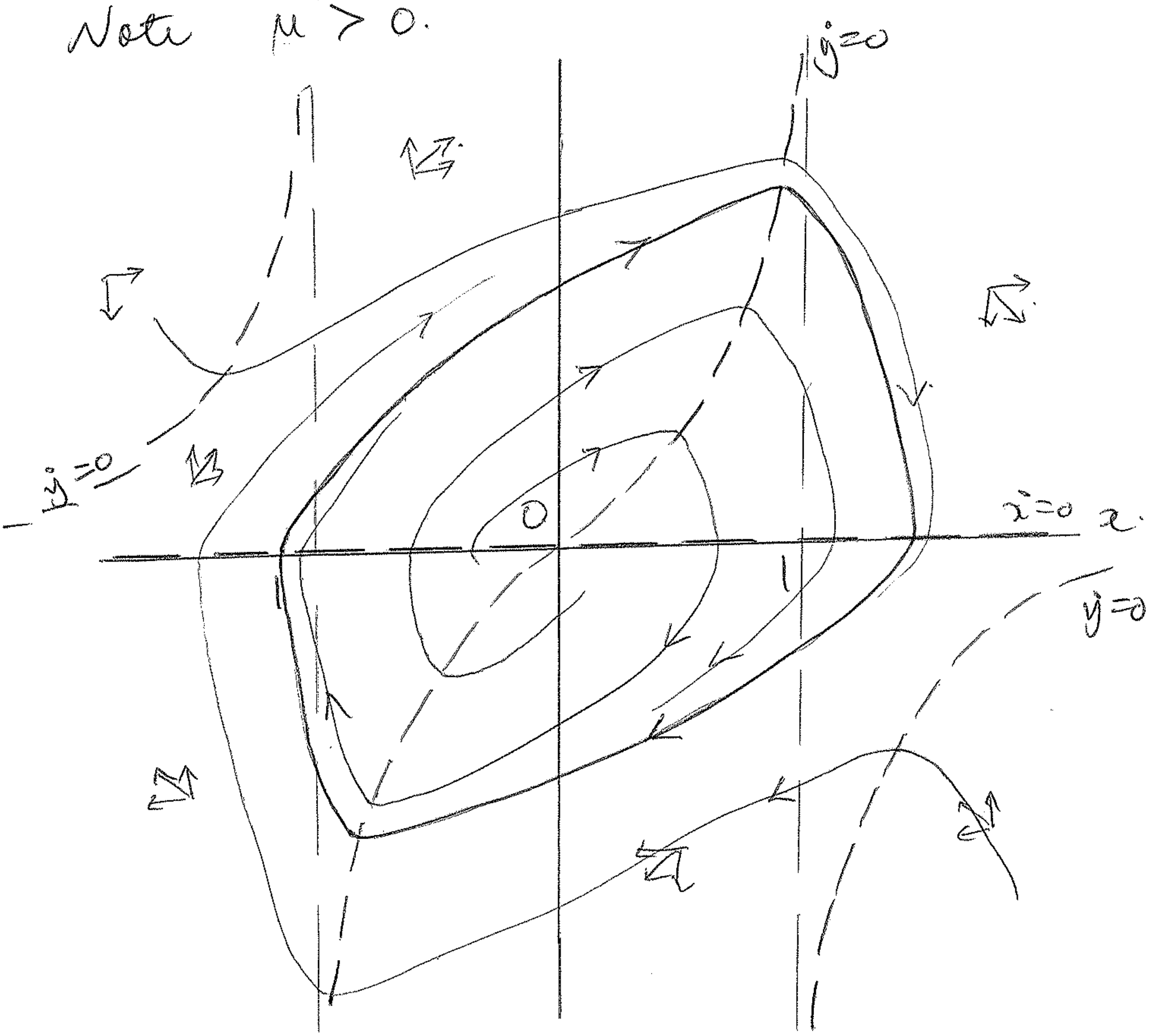
$\dot{y} = 0$ $y = \frac{x}{\mu(1-x^2)}$.

Flow $\dot{x} \gtrless 0$ if $y \gtrless 0$

$\dot{y} \gtrless 0$ if $y \gtrless \frac{x}{\mu(1-x^2)}$ and $|x| < 1$.

$\dot{y} \gtrless 0$ if $y \lessgtr \frac{x}{\mu(1-x^2)}$ and $|x| > 1$

Note $\mu > 0$.



3(i) $J = \begin{pmatrix} \mu - 3x^2 & 1 \\ -1 & \mu - 3y^2 \end{pmatrix}$

$J(0,0) = \begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}$.

Characteristic eqn $\lambda^2 - 2\mu\lambda + (1+\mu^2) = 0$.

$$\lambda = \frac{2\mu \pm \sqrt{4\mu^2 - 4(1+\mu^2)}}{2}$$

Eigenvalues are: $\lambda = \mu \pm i$

(ii) At Hopf bifurcation $Re(\lambda) = 0$ so $\mu = 0$

$\frac{d}{d\mu}(Re(\lambda)) = 1 \neq 0$ so there occurs a Hopf bifurcation at $\mu = 0$.

(iii) $V(x,y) = x^2 + y^2 \geq 0$ for all x, and y in a nhd of $(0,0)$.

$$\frac{dV}{dt} = 2x(\mu x + y - x^3) + 2y(-x + \mu y - y^3).$$

$$= 2\mu x^2 + 2xy - 2x^4 - 2xy + 2\mu y^2 - y^4.$$

$$= 2\mu(x^2+y^2) - 2(x^4+y^4)$$

< 0 for all $\mu \leq 0$.

Hence $V(x, y)$ is a Liapunov function for $(0, 0)$ when $\mu = 0$ and the steady state is asymptotically stable when $\mu = 0$.

(iv) If the steady state is asymptotically stable at Hopf bifurcation, then the bifurcating limit cycle is also asymptotically stable.