MATH 3063 – Tutorial Week 3 - Solns.

1. Let $x = \frac{N}{k}$ so $N = kx$

$\tau = tr$ so $\frac{d}{dt} = r\frac{d}{d\tau}$.

Then eqn becomes

$$kr\frac{dx}{d\tau} = rkx(1-x) - Ekx$$

or $$\frac{dx}{d\tau} = x(1-x) - \frac{E}{r}x.$$

Set $\mu = E/r$ to get $\frac{dx}{d\tau} = x(1-x) - \mu x = f(x)$.

At steady state $\frac{dx}{d\tau} = 0$ so

$$x^*(1-x^*) - \mu x^* = 0$$

so $x^* = 0$ or $1 - \mu - x^* = 0$.

$x^* = 1-\mu$. provided

$x^* \geq 0$ so $\mu \leq 1$.

Linear analysis

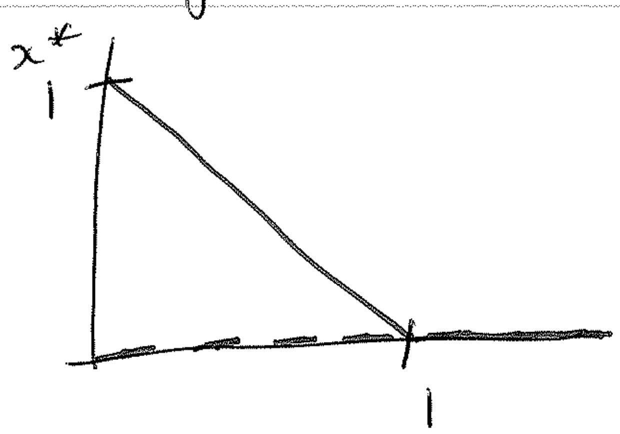
$$f'(x) = 1 - 2x - \mu.$$

At $x^* = 0$ $f'(0) = 1 - \mu \begin{cases} > 0 \text{ if } \mu < 1 & \text{unstable} \\ < 0 \text{ if } \mu > 1 & \text{stable} \end{cases}$

At $x^* = 1-\mu$ $f'(1-\mu) = 1 - 2 + 2\mu - \mu = \mu - 1 < 0$ if $\mu < 1$.

So when $x^* = 1-\mu$ exists, it is stable.

Bifurcation diagram:



(4) At steady state $\frac{dx}{dt} = 0$ so

$$x(\mu - x^2)(x - \mu + 2) = 0$$

Hence $x = 0$, $x = \pm\sqrt{\mu}$ if $\mu \geq 0$, $x = \mu - 2$.

Bifurcation occurs at $\mu = 0$.

$$f(x) = x(\mu - x^2)(x - \mu + 2)$$

$$f'(x) = (\mu - x^2)(x - \mu + 2) - 2x^2(x - \mu + 2) + x(\mu - x^2).$$

$$f'(0) = \mu(2-\mu) \begin{cases} < 0 & \text{if } \mu < 0 \text{ or } \mu > 2 \text{ stable} \\ > 0 & \text{if } 0 < \mu < 2 \text{ unstable} \end{cases}$$

$$f'(\sqrt{\mu}) = -2\mu(\sqrt{\mu} - \mu + 2) \begin{cases} > 0 & \text{if } \mu > 4 \\ < 0 & \text{if } \mu < 4 \end{cases}$$

$$f'(-\sqrt{\mu}) = -2\mu\left(-\sqrt{\mu}-\mu+2\right) \begin{cases} >0 & \text{if } \mu>1 \\ <0 & \text{if } \mu<1 \end{cases}$$

$$f'(\mu-2) = (\mu-2)(\mu-(\mu-2)^2)$$

$$=(\mu-2)(\mu-\mu^2+4\mu-4)$$

$$=-(\mu-2)(\mu^2-5\mu+4)$$

$$=-(\mu-2)(\mu-4)(\mu-1).$$

so $f'(\mu-2)$ >0 if $\mu<1$. unstable

<0 if $1<\mu<2$ stable.

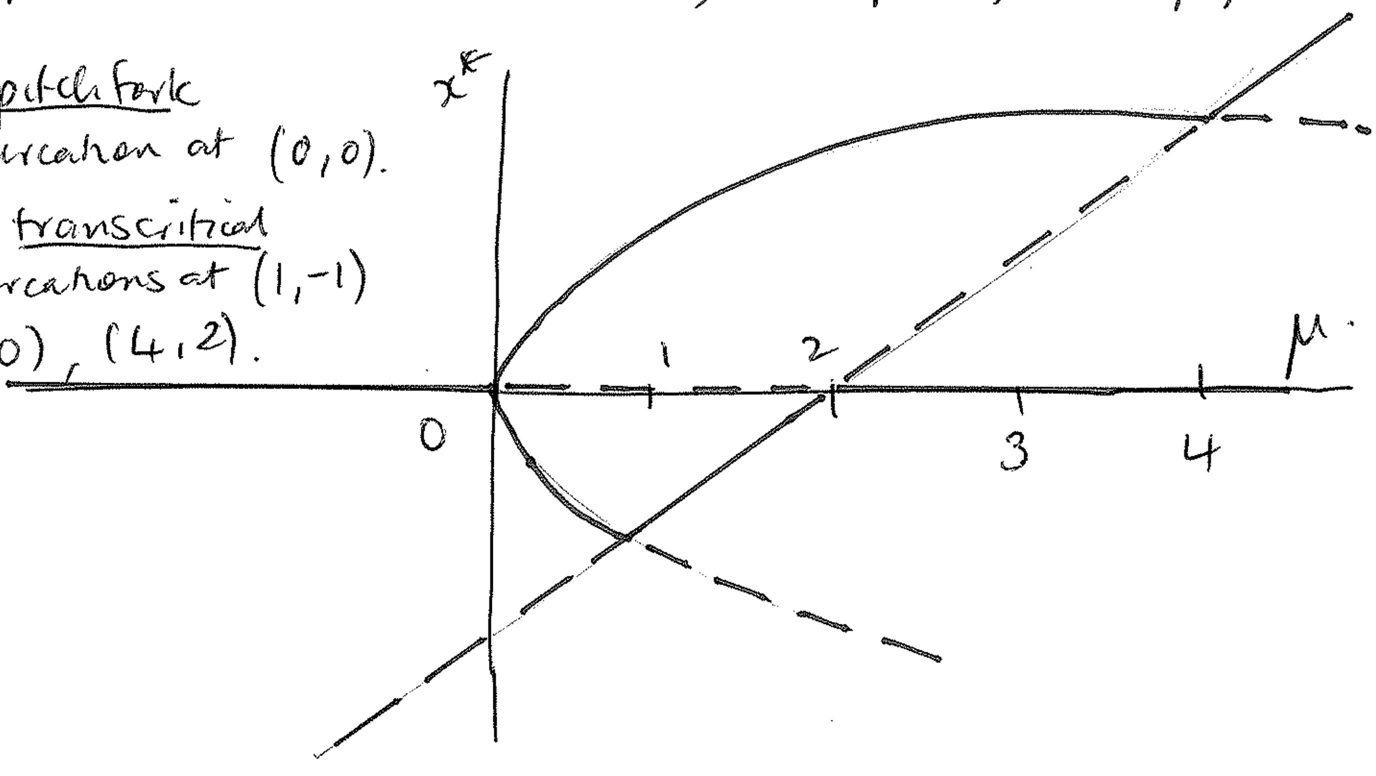
>0 if $2<\mu<4$. unstable

<0 if $\mu>4$. stable.

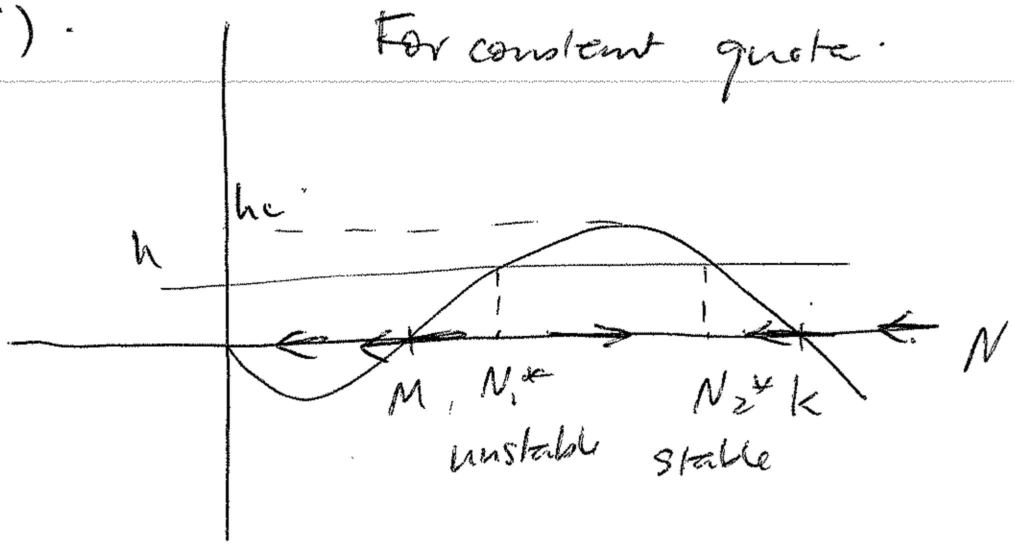
Bifurcation curves: $x^*=0$, $x^*=\mu-2$, $x^*=\mp\sqrt{\mu}$,

One pitchfork bifurcation at $(0,0)$.

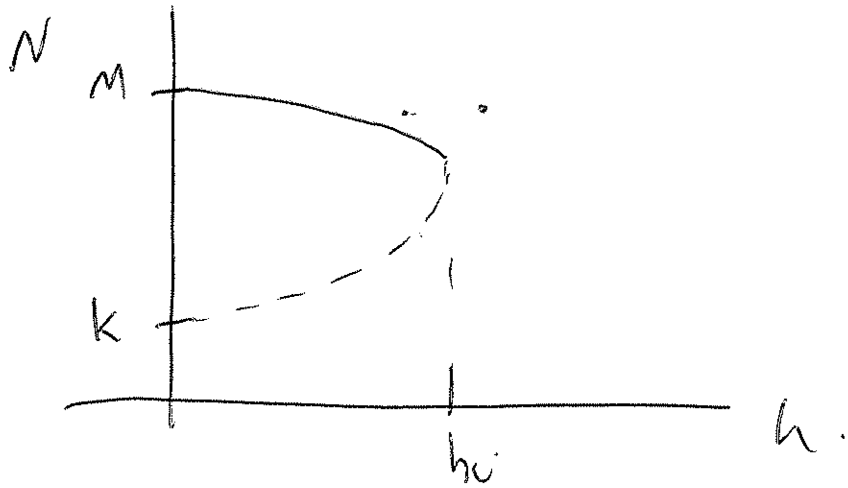
Three transcritical bifurcations at $(1,-1)$ $(2,0)$, $(4,2)$.



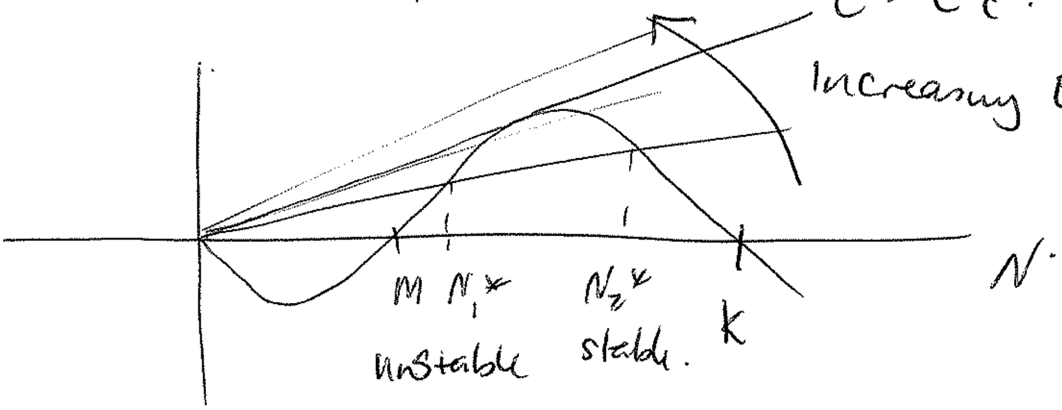
2. (i). For constant quota.



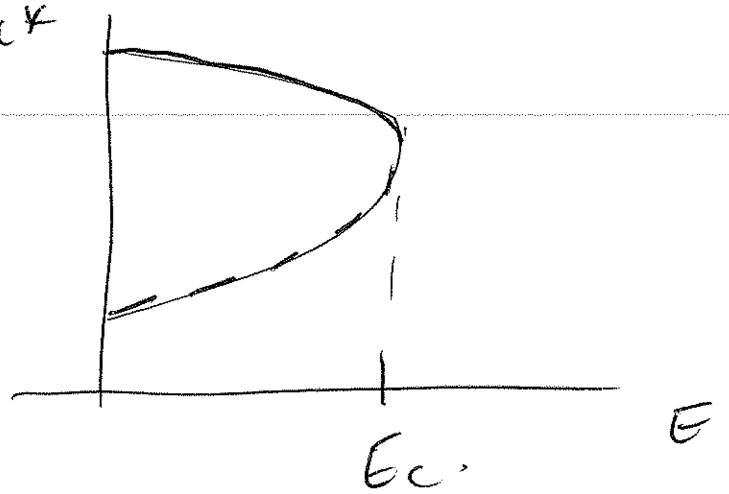
As h increases. N_1^* & N_2^* draw together.
When $h > h_c$ there are no steady states



(ii) For constant effort



Again there is a fold bifurcation at $E = E_c$.

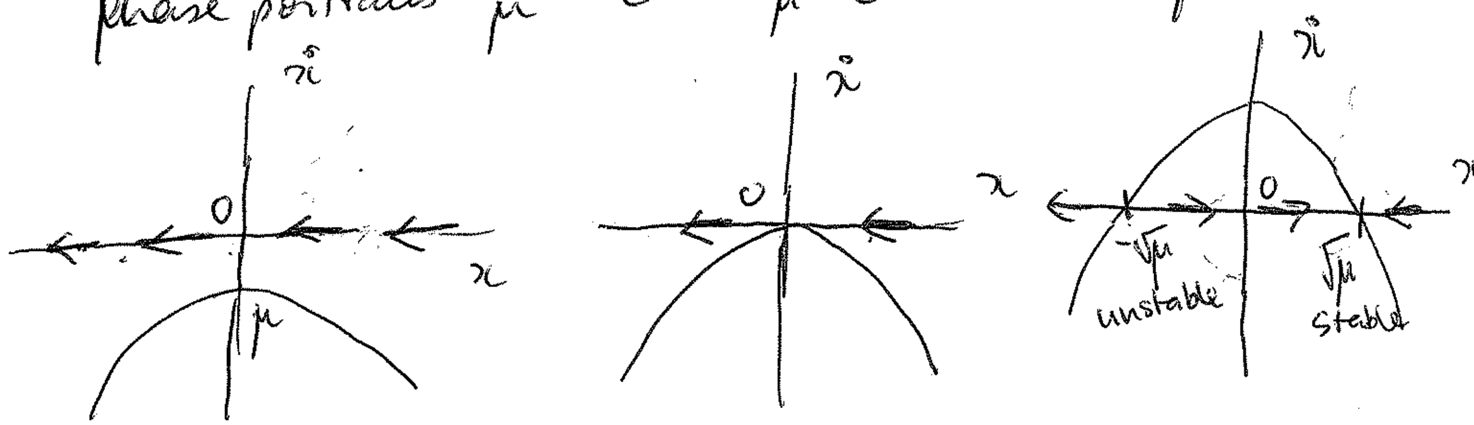


In both cases we have a fold bifurcation with a possible population crash.
When there is an Allee effect (as in this population) constant effort produces a popn crash — it does not go smoothly to zero. So harvesting this population ~~is~~ always has a risk of population crashes regardless of how its done.

3. Saddle-Node. ~~At~~ At steady state

$\frac{dx}{dt} = 0$ so $\mu = x^2$ $x = \pm\sqrt{\mu}$. Bifn at $\mu = 0$

Phase portraits: $\mu < 0$ $\mu = 0$ $\mu > 0$



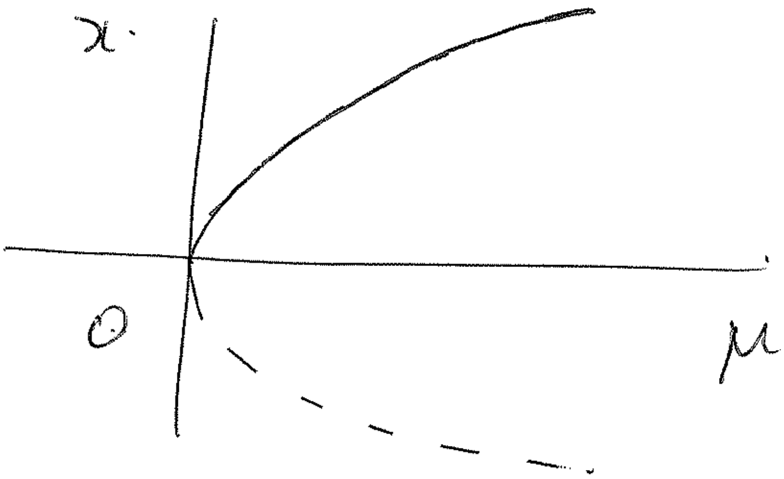
~~Near $\mu=0$~~

For $\mu \lesssim 0$ there are no steady states

At $\mu = 0$ there is one steady state of indeterminate stability

For $\mu \gtrsim 0$ there are two steady states

Bifurcation diagram.

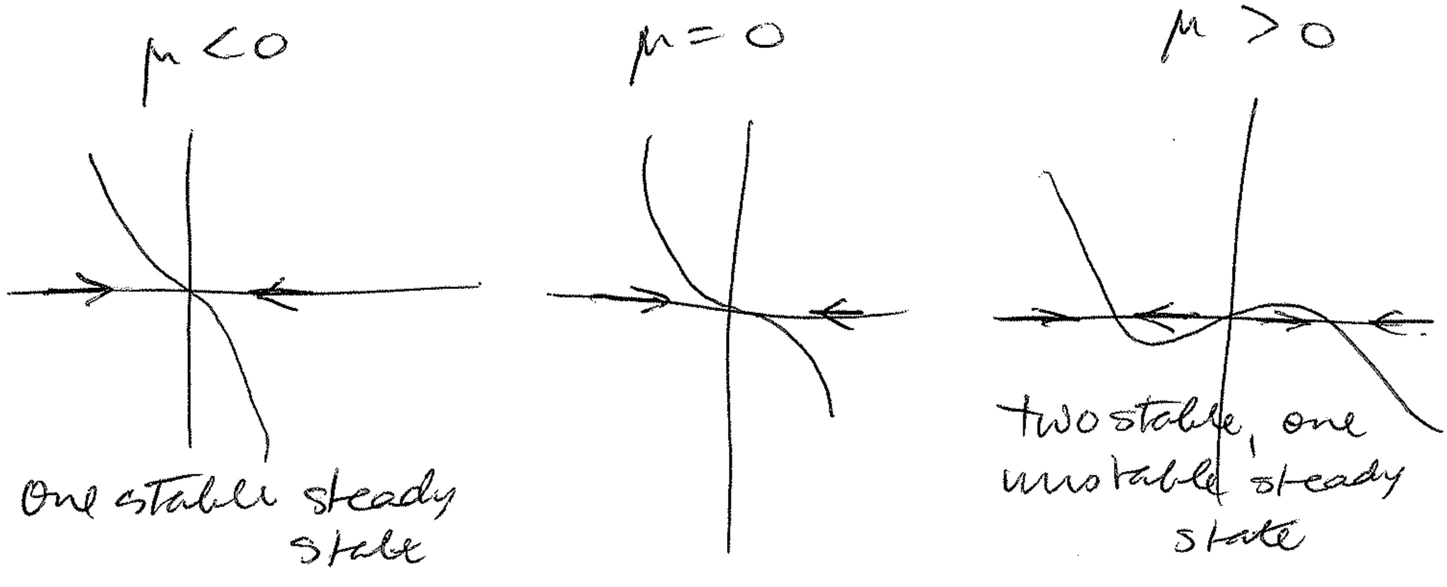


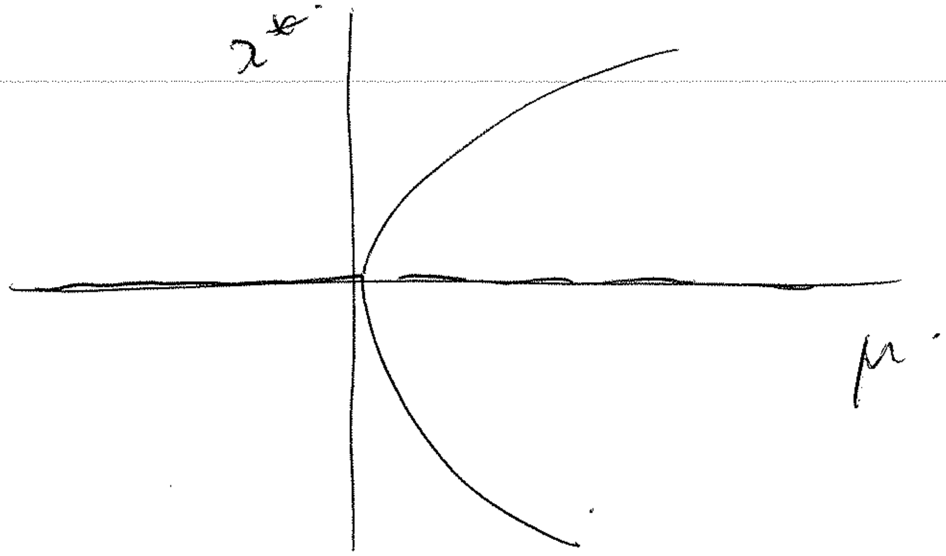
<u>Pitchfork</u> steady states $\mu x - x^3 = 0$ so

~~$\mu=0$~~ $x = 0$ or $x = \pm\sqrt{\mu}$ for $\mu \geq 0$.

Bif at $\mu = 0$

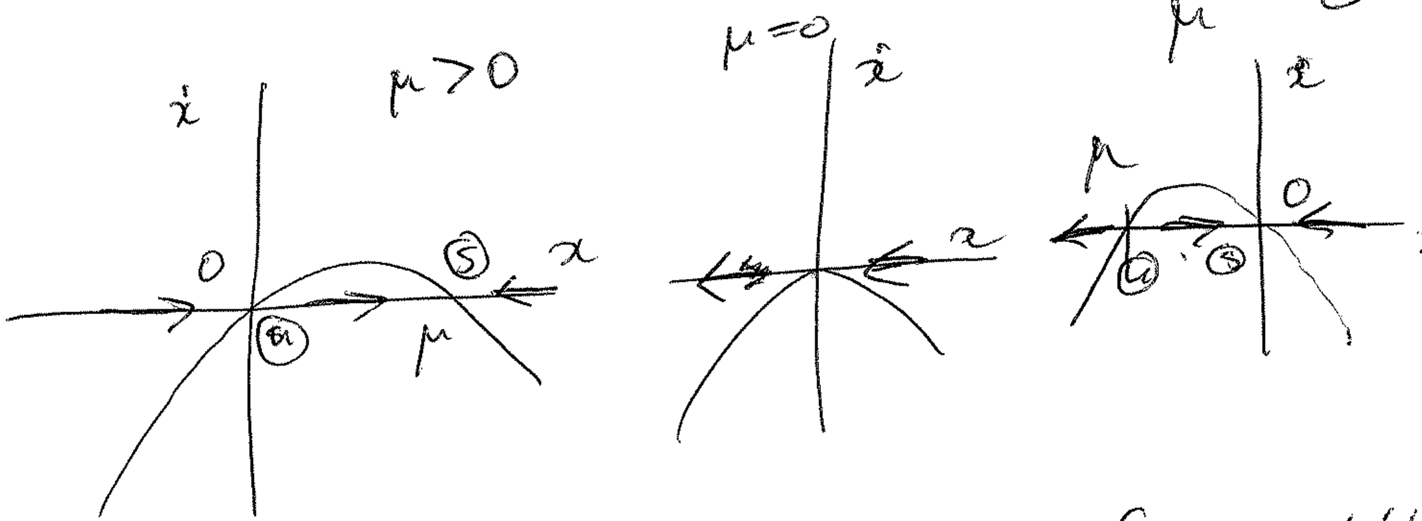
Phase portraits





Transcritical steady states at $(\mu - x)x = 0$

so $x = 0$ or $x = \mu$.



For $\mu \lesssim 0$ two ~~stable~~ steady states $\begin{cases} x = \mu \text{ unstable} \\ x = 0 \text{ stable} \end{cases}$.

At $\mu = 0$ one steady state of indeterminate stability

For $\mu \gtrsim 0$ two steady states $x = \mu$ stable, $x = 0$ unstable