THE UNIVERSITY OF SYDNEY

DIFFERENTIAL EQUATIONS & BIOMATHEMATICS

Semester 1	**Tutorial Week 9**	2011

1. For each of the following systems, find a suitable Liapunov function of the form $ax^2 + cy^2$, by finding suitable values of a and c and show that the steady state at $(0, 0)$ has the given stability.

(i) $\dot{x} = -x^3 + xy^2$, $\dot{y} = -2x^2y - y^3$, $(0, 0)$ is asymptotically stable.

(ii) $\dot{x} = -\frac{1}{2}x^3 + 2xy^2$, $\dot{y} = -3x^2y - y^3$, $(0, 0)$ is asymptotically stable.

(iii) $\dot{x} = x^3 - y^3$, $\dot{y} = 2xy^2 + 4x^2y + 2y^3$, $(0, 0)$ is unstable.

Adapted from *Elementary Differential Equations and Bounday Value Problems* 3rd edition by W.C. Boyce and R.C. DiPrima

2. Consider the following two sets of differential equations:

$$\frac{dx}{dt} = y - x(x^2 + y^2) \qquad \qquad \frac{dx}{dt} = y + x(x^2 + y^2)$$
$$\text{and}$$
$$\frac{dy}{dt} = -x - y(x^2 + y^2) \qquad \qquad \frac{dy}{dt} = -x + y(x^2 + y^2)$$

(i) For each system show that linear analysis classifies $(0, 0)$ as a centre.

(ii) Using the Liapunov functions $V(x, y) = U(x, y) = x^2 + y^2$ show that one system has a stable steady state at $(0, 0)$ and one has an unstable steady state at $(0, 0)$.

Adapted from an example in *Nonlinear Ordinary Differential Equations* by D.W. Jordan and P. Smith

3. Find a Liapunov function to establish the stability of the steady state $(0, 0)$ of the following equations. You may need to specify a neighbourhood of $(0, 0)$ where $\dot{V}$ is negative definite.

(i) $\dot{x} = -x + y - xy^2$; $\dot{y} = -2x - y - x^2y$

(ii) $\dot{x} = -x + xy^4$; $\dot{y} = -y^3 + y^4$

(iii) $\dot{x} = x + 4y$; $\dot{y} = -2x - 5y$

4. Find a Liapunov function to establish the *instability* of the steady state $(0, 0)$ of the following equations:

(i) $\dot{x} = 2x + y + xy$; $\dot{y} = x - 2y + x^2 + y^2$

(ii) $\dot{x} = x^2 - y^2$; $\dot{y} = -2xy$ Hint: try $U(x, y) = \alpha xy^2 + \beta x^3$ for suitable α and β.

Questions 3 and 4 adapted from *Nonlinear Ordinary Differential Equations* by D.W. Jordan and P. Smith