THE UNIVERSITY OF SYDNEY

DIFFERENTIAL EQUATIONS & BIOMATHEMATICS

Semester 1	**Tutorial Week 11**	2009

1. Sketch phase portraits consistent with the following information:

(*i*) an unstable limit cycle and three equilibrium states (one saddle and two stable nodes);

(*ii*) a stable focus and two limit cycles, one stable and one unstable.

2. (*i*) Show that the polar form of the nonlinear system

$$\dot{x} = -y + x(3 - 4\sqrt{x^2+y^2} + x^2 + y^2), \qquad \dot{y} = x + y(3 - 4\sqrt{x^2+y^2} + x^2 + y^2)$$

is given by

$$\dot{r} = r(1-r)(3-r), \qquad \dot{\theta} = 1.$$

(*ii*) Sketch the phase portrait for $\dot{r} = r(1-r)(3-r)$. Identify where $\dot{r} = 0$ and determine the stability of these steady states.

(*iii*) Hence sketch the phase plane for the system of equation. How many limit cycles are there? What is the stability of each limit cycle?

3. For each of the following autonomous systems, expressed in polar coordinates, determine all periodic solutions (limit cycles) and discuss their stability. (It may be helpful to draw a phase portrait for the $\dot{r}$ equation.)

(*i*) $\dot{r} = r(1-r)(r-2)$, $\dot{\theta} = 1$

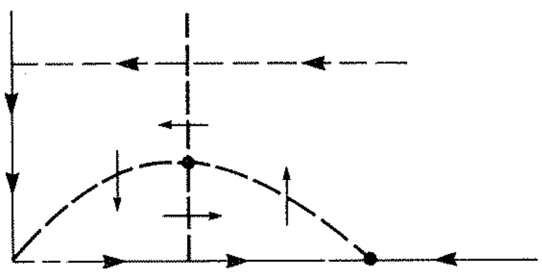
(*ii*) $\dot{r} = r(1-r)^2$, $\dot{\theta} = -1$

(*iii*) $\dot{r} = \sin \pi r$, $\dot{\theta} = 1$

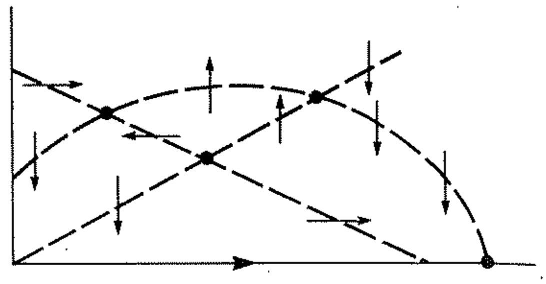
(*iv*) $\dot{r} = r|r-2|(r-3)$, $\dot{\theta} = -1$

Adapted from *Elementary Differential Equations and Bounday Value Problems* 3rd edition by W.C. Boyce and R.C. DiPrima

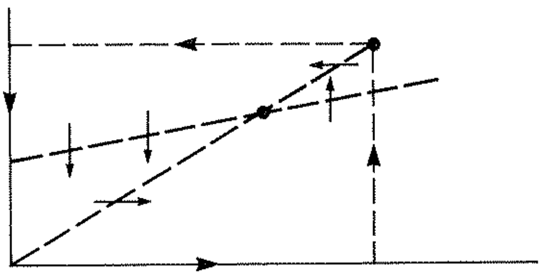
4. The nullclines are sketched on the phase planes below and the direction of the phase flow is indicated using arrows. The steady states, indicated by a large dot is unstable. In each case determine if the Poincaré-Bendixson Theorem can be used to prove the existence of a stable limit cycle.



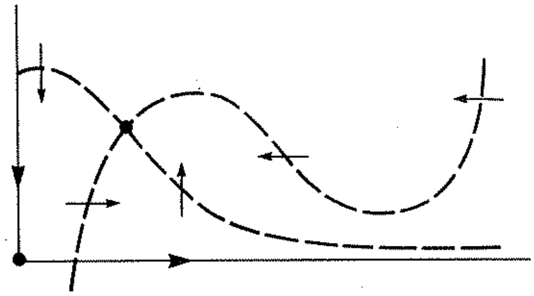
(a)



(b)



(c)



(d)

This question and diagrams have been adapted from *Mathematical Models in Biology* by Leah Edelstein-Keshet.