THE UNIVERSITY OF SYDNEY

Differential Equations & Biomathematics

Semester 1	**Tutorial Week 12**	2011

1. Consider the system

$$\frac{dx}{dt} = 1 - 3x + ax^2y$$
$$\frac{dy}{dt} = 2x - ax^2y$$

where $x \geq 0$, $y \geq 0$ and $a > 0$ is a parameter of the system.

(i) Show that this system has only one steady state.

(ii) Show that this steady state undergoes a Hopf bifurcation when $a = 1$. For what values of a is the steady state unstable?

(iii) The bifurcating limit cycle is always stable for this system. Sketch the phase plane when the limit cycle exists.

2. The Van der Pol oscillator is described by the equation

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0$$

where x can take on any real value and μ is a parameter which also can take on any real value.

(i) Show that this second order differential equation can be written as a set of two first order differential equations:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = \mu(1 - x^2)y - x$$

(ii) Show that this system has one steady state which undergoes a Hopf bifurcation when $\mu = 0$.

(iii) Sketch the phase plane when $\mu > 0$. (Note: the bifurcating limit cycle is stable in this case.)

3. Consider the system of differential equations:

$$\dot{x} = \mu x + y - x^3$$
$$\dot{y} = -x + \mu y - y^3$$

parameterised by μ.

(i) Use linear analysis to find the nature of the steady state of the system, $(0, 0)$. What are the eigenvalues of the Jacobian at $(0, 0)$ when $\mu = 0$?

(ii) Show that the system undergoes a Hopf bifurcation at $\mu = 0$.

(iii) Show that for $\mu \leq 0$, $V(x, y) = x^2 + y^2$ is a Liapunov function for the steady state $(0, 0)$ and hence show that $(0, 0)$ is asymptotically stable when $\mu = 0$.

(iv) Comment on the stability of the limit cycle that bifurcates from $(0, 0)$ at the Hopf bifurcation.