Semester 1	**Tutorial Week 13**	2011

1. The FitzHugh-Nagumo model for nerve conduction is given by

$$\frac{dv}{dt} = -v(v-a)(v-1) - w + I$$
$$\frac{dw}{dt} = \epsilon(v-w)$$

where $0 < \epsilon \ll 1$ and $0 < a < 1$.

(i) Show that when $I = 0$ there is a single steady state at $(0, 0)$. Use linear analysis to show that this steady state is stable. Sketch the phase plane.

(ii) Assume that $I > 0$ and sufficiently large that the steady state (v^*, w^*) lies on the middle branch of the cubic; that is, where the slope of the $\dot{v} = 0$ nullcline is positive. The equation of this nullicline is $w = f(v) + I$ where $f(v) = -v(v-a)(v-1)$ and so $f'(v^*) > 0$. Show that the Jacobian at (v^*, w^*) can be written as

$$J = \begin{pmatrix} f'(v^*) & -1 \\ \epsilon & -\epsilon \end{pmatrix}.$$

It can be shown that $f'(v^*) < 1$ for (v^*, w^*) on the middle branch of the cubic. Use this fact to show that (v^*, w^*) is unstable for ϵ sufficiently small. Sketch the phase plane in this case. Indicate the limit cycle which corresponds to a relaxation oscillation.

2. Consider the equations

$$\frac{dx}{dt} = -x^3 + \mu x - y$$
$$\frac{dy}{dt} = \epsilon x$$

where $0 < \epsilon \ll 1$ and μ is a parameter which can vary.

(i) Linearise the equation in the neighbourhood of the equilbrium point $(0, 0)$ and find the eigenvalues of the Jacobian matrix J.

(ii) Find the range of values of μ for which $(0, 0)$ is (a) a stable focus and (b) an unstable focus.

(iii) Assuming asymptotic stability of $(0, 0)$ when $\mu = 0$, show that the system undergoes a Hopf bifurcation at $\mu = 0$ to a stable limit cycle for $\mu > 0$.

(iv) Sketch the phase portraits for the two cases $\mu < 0$ and $\mu > 0$ and hence indicate how a stable limit cycle can exist in one case but not in the other.

3. Consider this alternative form of the FitzHugh-Nagumo equations:

$$\begin{aligned}\frac{dx}{dt} &= x - \frac{x^3}{3} - y + I \\ \frac{dy}{dt} &= \epsilon(x + a - by)\end{aligned}$$

where $a = 0.7$, $b = 0.8$ and $0 < \epsilon \ll 0$.

(i) For the case $I = 0$ draw the nullclines and sketch the flow in the phase plane. (The equilibrium point is at $(x^*, y^*) \approx (-1.2, -0.62)$). Indicate on your diagram two starting points, one of which gives a full action potential and one of which gives only a small perturbation.

(ii) Draw the phase plane when $I > 0$ and sufficiently large for a relaxation oscillation to occur.