Solutions

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(i) \(W_2 = (p_1 \oplus p_2)\)

(ii) \(W_3 = ((p_1 \oplus p_2) \oplus p_3)\)

(iii) \(W_4 = ((p_1 \oplus p_2) \oplus p_3) \oplus p_4\)

If \(d(W_k) = 4k-3\) then \(d(W_{k+1}) = \|(W_k \oplus p_k)\) = \(d(W_k) + 4 = 4k-3 + 4 = 4(k+1) - 3\), establishing the inductive step, and result follows by induction.
1. (b) (iii) \( V(W_n) = V(P_n) = T \) if zero variables have value \( F \), and zero is even, starting an induction.

Suppose \( k \geq 1 \) and result holds for \( W_k \) (ind. hyp.).

Suppose \( V(W_{k+1}) = V(W_k \oplus P_{k+1}) = T \). Either

(i) \( V(W_k) = V(P_{k+1}) = T \), or

(ii) \( V(W_k) = V(P_{k+1}) = F \).

In case (i), no. of \( P_1, \ldots, P_k \) with value \( F \) is even, by ind. hyp., so no. of \( P_1, \ldots, P_{k+1} \) is also even.

In case (ii), ... \( P_1, \ldots, P_k \) is odd, by ind. hyp., so ... \( P_1, \ldots, P_{k+1} \) is even.

Since \( V(P_{k+1}) = F \), in both cases no. is even.

Suppose conversely that no. of \( P_1, \ldots, P_{k+1} \) with value \( F \) is even. Then \( V(W_k) = T \) iff no. of \( P_1, \ldots, P_k \) with value \( F \) is even, by ind. hyp., in which case \( V(P_{k+1}) = T \) (to keep even no. \( F \)'s), so

\( V(W_k \oplus P_{k+1}) = T \), i.e. \( V(W_{k+1}) = T \).

Result now follows by induction.
2. (c) \((P \rightarrow Q) \equiv P\)

\[
\begin{array}{ccc}
T & F & F \\
F & F & F \\
\end{array}
\]

Counterexample: \(P = T, Q = T\) and \(P = F, Q = F\).

3. (b) \((P = Q) \rightarrow P\) \rightarrow P

\[
\begin{array}{ccc}
T & F & F \\
F & F & F \\
\end{array}
\]

4 and 6 are a contradiction, so we leave it.

3. (c)

1. (1) \(~(P = Q)\) \(\neg A\)
2. (2) \(~P\) \(\neg A\)
2. (3) \(P \rightarrow Q\) \(-SI (x)\)
1, 2, 4 \((P = Q) \land \neg (P = Q)\) \(1, 2 \land I\)
1. (5) \(~P\) \(2, 4 \land A\)
1. (6) \(P\) \(5 \land D\)
4. (4) \(Q\) \(\neg A\)

5. (8) \(P \rightarrow Q\) \(+SI (y)\)
1, 7, 9 \((P = Q) \land (P \rightarrow Q)\) \(1, 8 \land I\)
1. (9) \(~Q\) \(7, 9 \land A\)
1. (11) \(P \land \neg Q\) \(6, 10 \land I\)

4. (d)

1. (1) \((P = Q) \rightarrow P\) \(A\)
2. (2) \(~P\) \(A\)
1. 2 \((P = Q)\) \(1 \land MT\)
1. 2, 4 \((P \land \neg Q)\) \(2, 4 \land E\)
1. 2, 5 \(P\) \(4 \land I\)
1. 2, 6 \((P \land \neg P)\) \(4, 5 \land I\)
1. (7) \(~P\) \(5 \land A\)
1. (8) \(P\) \(2 \land RAA\)
4. 7 \(P\)
4. (9) \((P = Q) \land P\) \(1, 8 \land CP\)
3. (a) (i) 1 (1) \( \exists n \) \( F(n) \land G(n) \) \( A \)
2 (2) \( f(a) \land G(a) \) \( A \)
2 (3) \( f(a) \) \( 2 \land E \)
2 (4) \( \exists n \) \( F(n) \) \( 3 \land E \)
2 (5) \( G(a) \) \( 2 \land E \)
2 (6) \( \exists n \) \( C(n) \) \( 5 \land E \)
2 (7) \( \exists n \) \( f(n) \land \forall \exists m \) \( A(n) \land \exists m \land C(m) \) \( 4, 6 \land T \)
1 (8) \( \exists n \) \( f(n) \land \exists m \) \( A(n) \land \exists m \land C(m) \) \( 1, 2, 7 \land \exists E \)

(i) 1 (1) \( \forall n \) \( f(n) \lor \forall y \) \( C(y) \) \( A \)
2 (2) \( f(a) \) \( \forall E \)
2 (3) \( f(a) \lor C(a) \) \( 3 \land T \)
2 (4) \( \forall m \) \( f(m) \lor C(m) \) \( 4 \land T \)
6 (5) \( \forall y \) \( C(y) \) \( 6 \land E \)
6 (6) \( C(a) \) \( 7 \land T \)
6 (8) \( f(a) \lor C(a) \) \( 8 \land T \)
6 (9) \( \forall m \) \( f(m) \lor C(m) \) \( 1, 2, 5, 6, 9 \land \forall E \)
1 (10) \( f(a) \) \( \forall E \)

(ii) (b) Finally as (i) involved "a", used at (3).

All other steps are valid.

(b) (i) \( \mathbb{N} = \mathbb{Z}^+ \), \( F(n) = \text{"n is even"} \), \( G(n) = \text{"n is odd"} \)

All integers are even or odd, but not true that either all integers
are even or all integers are odd.

(ii) \( \mathbb{N} = \mathbb{Z}^+ \), \( P(n) = \text{"n+1"} \), \( Q(n) = \text{"n=0"} \), \( P(2) = ?(2) \)

i) True since \( P(2) \) is false, so \( \exists n \) \( P(n) = Q(n) \) is true.

However \( Q(y) \) is false for all \( y > \mathbb{Z}^+ \), so \( \exists y \) \( Q(y) \) is false,
but \( P(1) \) is true so \( \exists m \) \( P(n) \) is true, so \( \exists n \) \( P(n) = \exists y \) \( Q(y) \) is false.
(b) \( \mathbb{N} = \mathbb{Z}^+ \), \( R(x,y) \equiv x < y \). For all \( x \in \mathbb{Z}^+ \),

\( x \neq n \) and \( x \neq n+j \) so \( \mathbb{N}_1 \) and \( \mathbb{N}_2 \) hold.

(c) Let \( \mathcal{U} \) be a model for \( \mathbb{W}_1, \mathbb{W}_2, \mathbb{W}_3 \).

Choose \( x \in \mathcal{U} \) since \( \mathcal{U} \neq \emptyset \). Suppose, w.l.o.g.,

that \( x_1, \ldots, x_m \in \mathcal{U} \) all distinct and that

9) \( R(x_1, x_2), \ldots, R(x_{k-1}, x_k) \). By \( \mathbb{W}_3 \), there exists

\( \mathcal{U} \) such that \( R(x_k, x_{n+k}) \). If \( x_{k+n} = x \) for

some \( k \leq k \), then \( R(x_{k+n}, x_{k+n}) \), \ldots, \( R(x_k, x_{k+n}) \). So let

\( R(x_k, x_{k+n}) \), by repeated application of \( \mathbb{W}_3 \), essentially

\( \mathcal{U} \) (since \( x_1, \ldots, x_n \) are distinct and \( 2(x,n), \ldots, 2(x,n) \))

By induction, there are infinitely many distinct elements \( x, \ldots, x \). Since
5. (a) START

\[ R \xrightarrow{b} R \]
\[ R \xrightarrow{a} q_2 \]
\[ b \xrightarrow{1} b \]
\[ c \xrightarrow{g_0} \text{HALT} \]

M, always halts producing

\[ \begin{cases} \text{blank tape \& number at } A \text{'s left} & \text{even} \\ \frac{1}{2} & \text{if } \cdots A' \text{ odd} \end{cases} \]

(b) \[ X = \{ q_1 \rightarrow Rq_2, q_1 \rightarrow Rq_1, q_2 \rightarrow XuRq_1, q_2 \rightarrow XuRq_2, q_3 \rightarrow LuRq_3, q_3 \rightarrow LuRq_4, q_3 \rightarrow XuRq_3, q_4 \rightarrow Luq_4, q_4 \rightarrow LuRq_4, q_4 \rightarrow XuRq_3 \} \]

\[ \text{forever moving to right} \]

On gets stuck in state q_1 reading a blank.

3) If starting tape has an even number of 1's,

after rewriting them all with X's.

If starting tape has an odd number of 1's,

Then M, first replaces them with X's then

replaces the X's with A's doubling up 1's to the right

finishing with 2n 1's halting at cell past first.
Suppose to the contrary that \( M \) halts after receiving an input tape \( S \) with no symbol \( X \) occurring. Suppose \( M \) executes \( K \) instructions before halting, so the real-write head cannot move more than \( K \) squares to the right at the initial starting point. Add an \( X \) on a square located more than \( K \) squares to the right to form a new input \( S' \).

Since \( M \) only reads squares within \( K \) steps of the starting point, it will behave identically with \( S' \) as with \( S \), so will halt not reading an \( X \) except that \( X \) successfully writes an \( X \) when it exists. Hence \( M \) cannot halt without...
(a) For \( b \in B \), choose \( g(b) = \{ a \in A \mid f(a) > b \} \), which is nonempty since \( f \) is surjective. If \( b_1, b_2 \in B \) and \( g(b_1) = g(b_2) \), then by choice \( f(g(b_1)) = f(g(b_2)) = b_1 = b_2 \), verifying that \( g \) is injective.

(b) Both rules become unambiguous by only considering decimal expansions that avoid repeating 9's, with the exception of \( 1 = 0.9 \).

Let \( x = 0.d_1d_2\ldots \), \( y = 0.e_1e_2\ldots \), and \( f(x) = f(y) \).

If \( x = 0 \) then \( f(x) = f(0) = 0 \), and \( y = 0 = x \).

Suppose \( x \neq 0 \), and let \( d_i \) be any nonzero digit. Then \( d_i \times 10^i = f(n) = f(g) \), so \( d_i \times 10^i = e_j \times 10^i \) for some \( e_j \neq 0 \), so \( d_i = e_j \times 10^{-i} \), giving \( 10^i = 1 \), since \( d_i \in \{1,\ldots,9\} \), where \( i = j \) and \( d_i = e_i \). Similarly, if \( e_i \) is any nonzero digit, we get \( d_i = e_i \), proving \( x = y \). This verifies that \( f \) is injective.
(b) \[ h(x) = \{ i \in 2^+ \mid d_i = 0 \} = \{ i + 2^+ \mid i + 1 \} = Y, \]

verifying that \( h \) is surjective.

(c) We have \( f : [0, 1] \to 2^+ \) injective and 
\( h : [0, 1] \to 2^+ \) surjective, so there exists 
\( g : [0, 1] \to 2^+ \) injective, by (a). Hence, by 
Schönflies-Bernstein, there exist a bijection \( [0, 1] \to 2^+ \)
so \([0, 1]\) and \(2^+\) have the same cardinality.

(d) Clearly \( x \in \mathcal{P}(x) \), so \( \varphi(x) \in X \). If \( \varphi(x) \notin Y \)
then, by def., \( \varphi(x) \notin Y \), a contradiction.

Hence \( \varphi(x) \in Y \), so, by def., \( \varphi(x) = \varphi(y_0) \)
for some \( y_0 \in \mathcal{P}(x) \) such that \( \varphi(y_0) \notin Y_0 \).

If \( y = y_0 \) then \( \varphi(y_0) = \varphi(y) \in Y = Y_0 \), \( y \)
\( \varphi(y_0) \notin Y_0 \), a contradiction, hence \( y \notin Y_0 \).
(c) Suppose the collection $S$ of all sets $\mathcal{C}$ is a set, and form $P(S)$. Then elements of $P(S)$ are sets, so one element of $S$.

Hence $P(S) \subseteq S$. In the inclusion map $\varphi : P(S) \rightarrow S$, where $\varphi(x) = x$, it is injective. But $\varphi(y) = \varphi(y_o)$ for some $y \neq y_o$, by part (d), where

$$ y = \{ x \in S \mid x \notin x \} $$

contradicting that $\varphi$ is injective.

Hence $S$ is not a set.