CANCELLED DUE TO RAIN

Apologies for any inconvenience

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11 am 8/3/2012

The intended lecture notes follow this.
I will be discussed in future lectures.
Carto's Theorem: $\mathbb{R}$ is uncountable.


The power set of a set:

Let $X$ be any set and put

$$\mathcal{P}(X) = \{ \text{subsets of } X \}$$

called the power set of $X$.

E.g. If $X = \{1, 2, 3\}$ then

$$\mathcal{P}(X) = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \}$$

If $X = \{1, 2, 3\}$ then $\mathcal{P}(X)$ looks like

[Diagram]

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[Diagram]
If \( |x| = n < \infty \) then \( |\Theta(x)| = 2^n \). (Why?)

What if \( x \) is infinite?

Claim: \( |\Theta(x)| > |x| \)

where \( |S| \) is called the cardinality of a set (not yet rigorously defined).

The claim is saying there exists an injective mapping \( x \to \Theta(x) \)

but no bijective mapping.

The claim makes precise the idea that the "infinity" of \( \Theta(x) \) is "greater" than the "infinity" of \( x \)
Proof: The map \( x \mapsto \mathcal{P}(x), x \mapsto \mathcal{P}(\mathcal{P}(x)) \) is clearly injective, so \( 1 \times 1 \leq (\mathcal{P}(x)) \).

Suppose there exists a bijection \( f: X \rightarrow \mathcal{P}(x) \).

Put \( y = \{ x \in X \mid x \notin f(x) \} \) (variation of Russell's Paradox).

Since \( f \) is onto, \( y = f(y) \) exists.

Either \( y \in y \) or \( y \notin y \) (Law of Excluded Middle).

If \( y \in y \) then \( y \notin f(y) = y \), by definition of \( y \), so \( y \notin y \) and \( y \notin y \).

If \( y \notin y \) then \( y \in f(y) \) so \( y \in y \), by definition of \( y \), so \( y \in y \) and \( y \notin y \).

Hence \( \otimes \) is false, so no bijection exists.
Hence \(|x| < |\mathcal{P}(x)|\)

In particular \(|\mathcal{P}(\mathbb{Z}^+) > |\mathbb{Z}^+|\).

But also \(|\mathcal{R}| > |\mathbb{Z}^+|\) (Cantor).

In fact

**Theorem:** \(|\mathcal{R}| = |\mathcal{P}(\mathbb{Z}^+)|\)

Post: sequence of difficult exercises.

**Continuum Hypothesis rephrased**

There is no cardinality strictly between \(|\mathbb{Z}^+|\) and \(|\mathcal{P}(\mathbb{Z}^+)|\).

**Generalised Continuum Hypothesis**

For any set \(X\), there is no cardinality strictly between \(|X|\) and \(|\mathcal{P}(X)|\).